NUMERICAL SIMULATIONS OF BLACK HOLE ACCRETION

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ABSTRACT. A resume of fruitful interaction between analytical and numerical approach to the subject of accretion disks around black holes is here presented. We review time dependent simulations of unviscous and moderately viscous accretion flows. Convection or pressure dominated flows admit subkeplerian solutions with or without shocks. The viscous adiabatic solutions, however, don't match smoothly to the keplerian disk solutions. We show also that many different type of mechanisms can trigger quasi periodic oscillations of the flow and consequently of the emitted radiation.

Key words: Accretion disks, shock waves; black holes

1. Introduction

Basic models of accretion disks assume a keplerian rotation of the flow of matter falling onto the compact astrophysical object (Pringle and Rees 1972, Shakura and Sunyaev 1973, Novikov and Thorne 1973). However, the structure of the flow, when the angular momentum is smaller than keplerian or when the advecttion and pressure term are required, is still far from being clear. Analytical solutions are difficult and often uncertain, due to *ad hoc* simplifications like: vertical equilibrium assuption, self similarity of the flow, neglecting of pressure and viscous terms, unappropriate boundary conditions and so on. Numerical solutions don't have general properties, but can verify the validity of the analytical solutions. Furthermore global stability studies of the analytical solutions are possible only by numerical simulations. They can even point to some new phenomenon as an *experimental* result. We will see some examples of this possibility. We will examine unviscous accretion flows and adiabatic viscous accretion solutions, comparing analytical steady solutions with time dependent numerical solutions obtained with different numerical algorithms. Section 2 describes the physical context and gives the relevant analytical equations to be solved. Section 3 gives the steady state solutions. Section 4 describes briefly the numerical methods. Section 5 resumes the more interesting results.

2. The physical scenario

Let us assume a viscous gas fall onto a black hole with an initial subkeplerian amount of angular momentum. This means that, apart the simple case of bremsstrhalung cooling, we assume there is no cooling in our gas. Despite this limitation, different regimes due to different values of the polytropic gas index $\gamma = c_p/c_v$ will appear. We have considered also the case of bremsstrahlung cooled gas, due to the possibility to make a simple treatment of the derived solutions and their relevant consequences. We assume that the gravitational forces are derived from the Paczyński Wita potential. This is enough accurate to point out the main physical processes and to produce the relativistic behaviour of the flow, avoiding the complications of an exact general relativistic treatment. Exact relativistic steady state solutions for accretion onto Kerr black holes have been discussed by Chakrabarti (1996). The basic equations to be satisfied are the following ones: Assume axis symmetric case, i.e. $\frac{\partial}{\partial\phi} = 0$.
The mass conservation equation

$$\frac{D\rho}{Dt} = -\rho\nabla\mathbf{v}$$

The angular momentum equation is:

$$\rho\frac{Dv_\phi}{Dt} + \rho\frac{v_\phi v_r}{r} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{r\phi}\right) + \frac{\tau_{r\phi}}{r}\right] + \frac{\partial\tau_{\phi z}}{\partial z}$$

The energy equation

$$\frac{D\epsilon}{Dt} = -\frac{P}{\rho}\nabla\mathbf{v} + \frac{\Phi}{\rho} + \Lambda$$

where ϵ is the thermal energy per unit mass, Φ is the dissipation function, $\Lambda \sim \rho T^{0.5}$ the cooling function and other symbols have the usual gas dynamic meaning.

The expressions of the stresses will be specified in the subsequent discussion. These are the equations that will be integrated by time dependent codes.

3. Steady state solutions

Let us put the previous general time dependent equations in the typical steady state form:

Mass conservation requires

$$\dot{m} \propto r\rho v_r Z_{disk} = const$$

Radial momentum

$$v_r \frac{dv_r}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - G\frac{M_*}{(r-r_g)^2} + \frac{\lambda^2}{r^3}$$

with λ the angular momentum per unit mass
Tangential momentum

$$\frac{\rho v_r}{r}\frac{d\lambda}{dr} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{r\phi}\right)$$

Vertical momentum

$$Z_{disk} \approx \sqrt{\frac{2v_s^2}{\gamma\left(\frac{GM_*}{r}\right)}}\,(r-r_g)$$

Energy equation

$$v_r \frac{d\epsilon}{dr} = -\frac{P}{\rho}\frac{1}{r}\frac{d(rv_r)}{dr} + \frac{\Phi}{\rho}$$

Viscosity prescription

$$\tau_{r\phi} = -\alpha P$$

or

$$\tau_{\phi r} = \mu r \frac{\partial \Omega}{\partial r}$$

with μ the dynamic viscosity coefficient, $r_g = 2GM_*/c^2$ the Schwartzschild radius of the stellar object and gravitational forces derived by Paczyński Wiita potential $\Psi(r) = -GM_*/(r-r_g)$.

An interesting point, apparently unnoticed in the literature, is the fact that it is possible to have a constant energy property reformulating the energy equation in this way :

$$div\left[\rho\mathbf{v}\left(\frac{1}{2}\mathbf{v}^2 + \Psi(r) + h\right) - \mathbf{v} : \overleftrightarrow{\tau}\right] = 0$$

here h is the enthalpy function $h = \epsilon + \frac{P}{\rho}$. This energy equation, after insertion of the stress definition, even in its differential form, and using the mass conservation equation, in general gives the following relationship :

$$\frac{1}{2}v_r^2 - G\frac{M_*}{(r-r_g)} + \frac{a^2}{(\gamma-1)} - \frac{\lambda^2}{2r^2} + \frac{\lambda\lambda_e}{r^2} = \frac{const}{\dot{m}}$$

Here λ_e is the angular momentum at the inner edge of the disk, where the tangential stress vanishes. Above formulae constitute the typical set of equations to be solved to obtain the steady state solutions.

3.1 Unviscous cases

The unviscous case is simple to be solved, for a full account see Chakrabarti 1990. Essentially it is very similar to Bondi problem (Bondi, 1952). Let us resume the solution for the pure 1D flow.

Mass conservation gives:

$$\dot{m} = \rho r v = \rho r M a$$

M is the mach number, $\dot{m}$ is the accretion rate, ρ is the density, a is the sound speed.

Since the flow is unviscous the total energy is constant, the Bernoulli theorem is valid:

$$\frac{1}{2}v^2 + \frac{a^2}{(\gamma-1)} + \frac{\lambda^2}{2r^2} - G\frac{M_*}{(r-r_g)} = E$$

To solve the system we add the politropic relation valid only for isentropic flow.

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{\frac{2}{(\gamma-1)}}$$

If we put together all terms we find:

$$\dot{m} = K \cdot f(M) \cdot A(E,\lambda,r)$$

with

$$f(M) = \frac{M}{\left(\frac{1}{2}M^2 + \frac{1}{(\gamma-1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

and

$$A(E,\lambda,r) = r \cdot [E - \Phi(\lambda,r)]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Φ is function only of λ and r . This solution is valid for any isentropic branch, the constant $K = K(a_0,\rho_0)$ is related to the entropy value.

The f function has a maximum at $M = 1$, the A function has in general two relative minima.To small energy constant E values correspond smaller A values of the minimum at larger distances from the BH. The other relative minimum is related to the λ value: large λ produce smaller A values close to the BH. The $f \cdot A$ product must be constant along a flow. Therefore a minimum of A has to correspond to the maximum of f .

A flow connecting very large distances to the BH horizon has $M \ll 1$ for large r and $M \gg 1$ close to the horizon. As A approaches the outer minimum value $A_{out} = A_{\min}(r_{out})$ the mach number has to increase to $M = 1$,i.e. the solution has one sonic point at r_{out} ; so we may easily find this kind of solutions solving the implicit equation for M(r) with an iteration procedure or with any commercial equations solver :

$$f(1) \cdot A_{out} = f(M) \cdot A(E,\lambda,r)$$

Let us call this solution $M_1(r)$.

We may also say that, if it exists a transonic solution starting subsonically from some region and finishing supersonically into the BH, it must have the same behavior of the previous solution. However now the sonic point has to correspond to the inner relative minimum of the A function $A_{inn} = A_{\min}(r_{inn})$; so this second solution come from this implicit equation:

$$f(1) \cdot A_{inn} = f(M) \cdot A(E, \lambda, r)$$

Let us call this solution $M_{sub}(r)$.

Now, if a shock on the solution $M_1(r)$ occurs it produces a post shock mach value M_2 given by the Hugoniot relations. If the M2 curve crosses the M_{sub} curve then a shock can , in principle, occur at that position if the M_{sub} solution correspond to an entropy status greater than the M_1 one.

The stable shock position is the outer one and is obtained solving:

$$M_2(r, E, \lambda) = M_{sub}(r, E, \lambda)$$

The solutions so obtained correspond exactly to the ones obtained integrating the differential equations in the way described by Chakrabarti in his book.

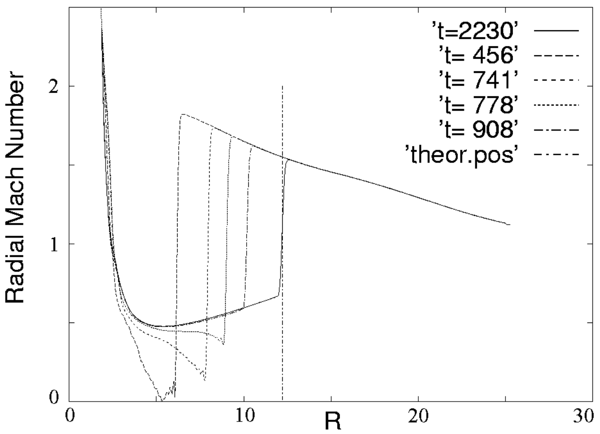

Figure 1: Mach number for steady state 1D solution and time dependent results.

Fig.1 shows the analytical solution with the overplotted numerical solutions obtained at different times. It can be seen the exact agreement between the two solutions for long integration time.

3.2 Viscous isothermal flows

This case mimics flows with very efficient cooling. It is a plain schematic case, but it is useful to learn the topology of the solutions. For the $\tau_{r\phi} = -\alpha P$ prescription it is even possible to have an algebraic solution (i.e. no differential equation). Combining the mass conservation, angular momentum and energy equations we are lead to solve the following algebraic equation:

$$\frac{1}{2}v_r^2 - K^2 \ln|v_r| - G\frac{M_*}{r - r_g} - K^2 \ln(r) - \frac{\lambda^2}{2r^2} + \frac{\lambda\lambda_e}{r^2} = B$$

with $\lambda = \lambda_e + \alpha\frac{K^2 r}{|v_r|}$ This fact allows us to prove a trivial, but interesting, fact: steady keplerian isothermal disks cannot exist, since the substitution of λ with λ_{kepler} leads to contradictory results for v_r, whatever be the initial conditions. However, it is possible to make a numerical simulation, constraining the gas flow to be isothermal, with small constant sound speed, and initial keplerian angular momentum to obtain a plain keplerian disk (cfr. Chakrabarti and Molteni, 1995). We may conclude that the keplerian disks are steady, but in an average way: they are affected by a turbulence (not the S.S. one) producing convection cells. The time dependent S.P.H. simulations agree perfectly with the analytical results obtained taking into account convection, viscosity and pressure terms. For a detailed discussion see Chakrabarti and Molteni (1995).

3.3 Viscous adiabatic cases

We adopt the basic assumptions valid for simple models of ADAFs (Narayan et al. 1998), i.e.: Low cooling efficiency, that is neglect of cooling terms, Shakura Sunyaev viscosity, vertical equilibrium, gradient of pressure and convective terms are retained in the equations. Let us briefly discuss the pure 1D case (i.e. $\frac{\partial}{\partial z} = 0$), whose results can be compared exactly with the time dependent numerical simulations. In the case of $\tau_{r\phi} = -\alpha P$ viscosity prescription we have the following set of equations (to make the formulae clean we are using v instead of v_r).

For mass conservation

$$\dot{m} = r\rho v$$

For the angular momentum

$$\lambda = \lambda_e - \alpha\frac{a^2}{\gamma v}r^2$$

For the radial momentum, retaining first order terms in α, after substitution of ρ P and λ from the above equations and use of the reference quantities r_g, and c (light speed) to adimensionalize the equation, we have

$$\frac{dv}{dr} = \frac{2\,\lambda_e a^2 \alpha + \left[-\left(ra^2 + \frac{\lambda_e^2}{r}\right) + \frac{r^2}{2(r-1)^2}\right] v}{r^2\left(a^2 - v^2\right)}$$

Energy equation gives

$$\frac{1}{2}v^2 + \frac{\lambda^2}{2r^2} - \frac{1}{2(r-1)} + \frac{a^2}{(\gamma - 1)} + \alpha\frac{a^2}{\gamma}\frac{\lambda}{r \cdot v} = E$$

where a is the adiabatic sound speed.

Therefore, we have four unknowns variables v, a, ρ, and λ that can be obtained solving the above set of equations, one differential and three algebraic. The radial momentum differential equation shows the presence of a sonic point: The r_s value for which $a(r_s) = -v(r_s)$.

For a regular accretion flow the derivative at r_s must be continuous , this requires that the numerator and the denominator of the equation be simultaneously zero at r_s and consequently the value of the derivative $\left(\frac{dv}{dr}\right)_s$ is well fixed. In this case the value of the derivative at the sonic point depends univocally on the values of α, E and λ_e.

So it is wrong to put its value to zero or whatever other value, as many other authors do (Narayan et al, 1997). Typical solutions of this case are shown in the paper by Chakrabarti 1998.

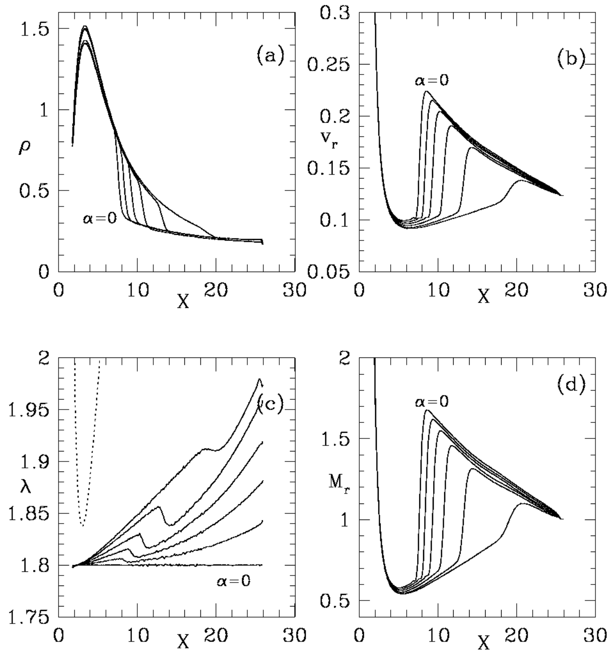

Figure 2: Solutions of 1D adiabatic viscous flows for increasing viscosity α parameter

Fig.2(a-d) show stationary shock locations as functions of Shakura-Sunyaev viscosity parameter α in thin, rotating, accreting flows. In (a), we plot density $\rho(r)$, in (b), we plot radial velocity $v_r(r)$, in (c), we plot $\lambda(r)$ and in (d), we plot Mach number $M_r(r)$. The α values for which the curves are drawn are (left to right in [a, b, d] and bottom to top in [c]): 0, 10^{-4}, 2×10^{-4}, 3×10^{-4}, 4×10^{-4} and 4.6×10^{-4} respectively.

4. The numerical algorithms

We exploited basically two kind of numerical algorithms to integrate the time dependent equations. One is a Smoothed Particles Hydrodynamics code (SPH) formulated in cylindrical coordinates. SPH is a lagrangean method based on interpolation criteria of the fluid variables and of their derivatives. The interpolation points move with the fluid speed. A full discussion of the method is given in Monaghan (1985). For its formulation in cilindric coordinates cfr. Molteni and Sponholz 1994. Another code based on the Total Variation Diminuishing (TVD) procedure have also been used. A comparison of the results of the two code has been performed (Molteni, Ryu and Chakrabarti 1996) confirming the reliability of the simulations results. SPH has been used also for viscous and cooled flows. For the study in the XY plane, using a r ϕ coordinates, another set of two different type of TVD codes has also been used cfr. Molteni, Toth and Kutznezov 1999, in this case SPH had too much shear numerical viscosity.

5. Time variability

Obviously the time dependent approach, by the use of numerical algorithms, allows the time variability study of the solutions. Indeed we found that there are many different ways in which the flow can exhibit time variability.

5.1 Time variability for unviscous cases

For unviscous ideal gas flow we find that axis symmetric 1D solutions are stable. However, if the axis symmetry is broken, then a very interesting phenomenon occurs. We performed simulations on the XY plane (no zeta motion $\frac{\partial}{\partial z} = 0$) of the axis symmetric shock, using a TVD code in r, ϕ coordinates that has a very low numerical viscosity (detailed results are give in Molteni et al. 1999). The shock forms at the predicted position and is stable, but if the flow is perturbed with a small (even 1% is enough) perturbation then the circular shape of the shock changes and the resulting deformation persists for ever, even if the average shock position is close to the original one. Therefore the radiation emitted by the post shock zone changes in time, producing Quasi Periodic Oscillations. The variations are more irregular for shocks produced with large angular momentum, while shocks with small angular momentum may produce quite regular oscillations. Fig.3 shows the isocontours of the radial mach value.

Unviscous solutions for ideal gas may produce oscillations also when the motion along the vertical direction is allowed. In this case, $\frac{\partial}{\partial z} \neq 0$, for low energy constant solutions, the disk thickness is very small and vertical compression and expansion make the solutions more unstable : recurrent shock formation within a finite range of the radial position occurs. Details of these simulations are given in Ryu et al. 1997.

Futher oscillations are possible when a cooling is present. We examined the plain case of an ideal gas with $\gamma = 5/3$ cooled by the bremstrahlung process. We considered therefore optically thin accreting regimes. In this case the oscillations occur when the fall time is close to the cooling time. We observed nearly regular oscillations in the 1D case. The centrifugal barrier

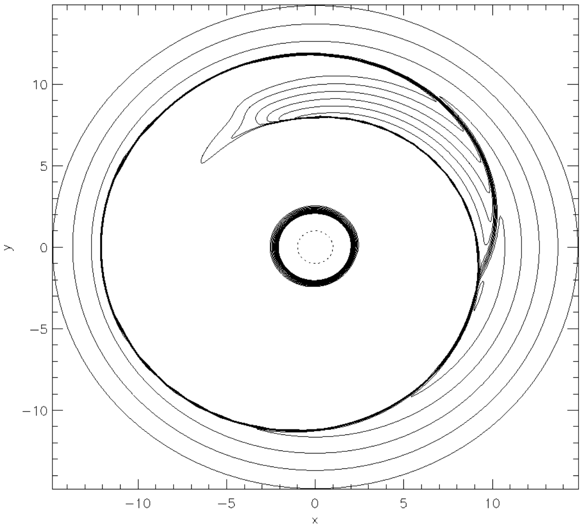

Figure 3: Radial Mach contour levels in the XY plane of a deformed, but permanent shock.

acts like a rigid wall. In the post shock region the gas reaches extremely high temperatures. In the 2D cases the post shock region is also very hot but now it can expand and collapse in the vertical direction producing a large hot corona around the thin accretion flow.

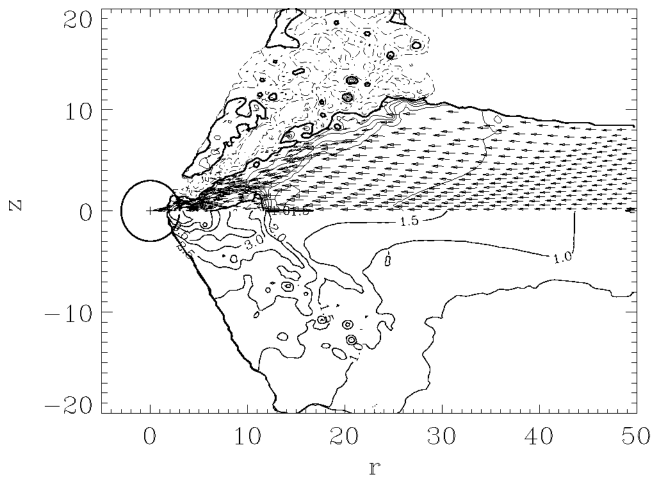

Figure 4: Hot corona of a unviscous accretion flow with shock cooled by bremsstrahlung.

Fig.4 shows the isodensity levels of the hot corona, with the velocity field overplotted, in the maximum expansion phase.

5.2 Time variability for viscous cases

In the case of adiabatic and viscous accretion, the flow in the subsonic region has large viscosity due to the higher post shock temperatures. In these cases the post shock flow forms an almost keplerian disk. The matter piles up in the disk and discharges into the black hole in a recurrent way. Details are given in the work by Lanzafame et al. 1998. In Fig.5(a-d), we show the first simulations of shocks in viscous flows where they oscillate periodically. In (a-c), roughly half of the cycle is shown where the shock location decreases monotonically. Total number of particles in this simulation is on an average around 10, 000 (see, Fig. 6 below). In (d), the shock drifted again outward. Note that apart from the axisymmetric shock oscillation, a new, corrugated instability, is also apparent in Fig. 4(d). Fig.6 shows the mass of the disk versus the elapsed time.

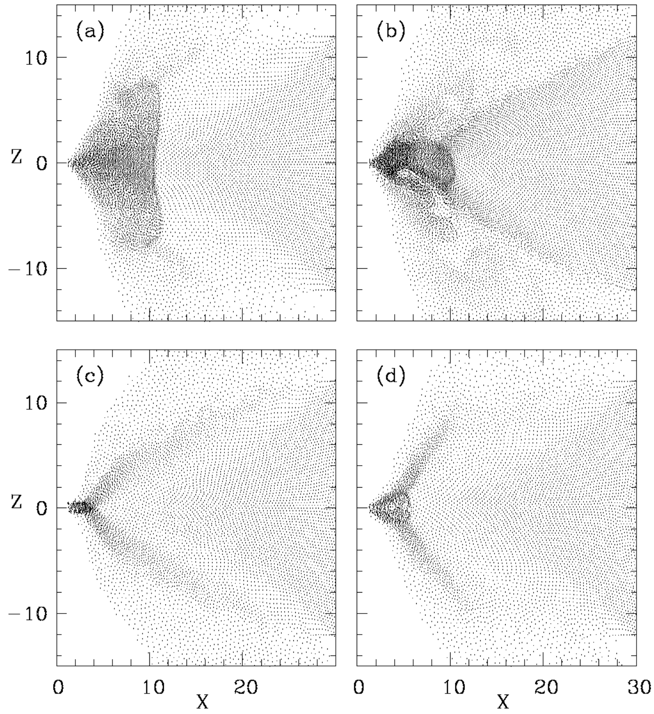

Figure 5: R-Z projection of the particles for viscous adiabatic flow with shock.

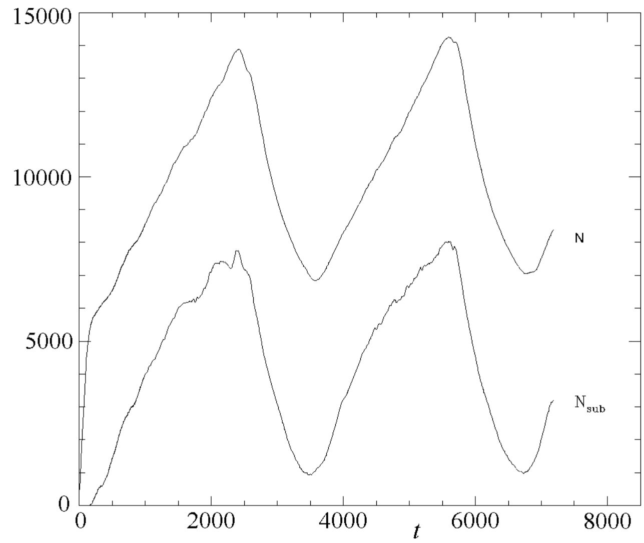

Figure 6. Mass versus time of the adiabatic viscous disk.

6. Conclusions

Numerical time dependent simulations of accretion flows demonstrate that analytical exact solutions can be obtained only if the sonic point conditions are treated appropriately. Furthermore the *numerical experiments* have shown that finite amplitude oscillations of the flow can be produced in many different ways. Consequently the time variability and the Quasi Periodic Oscillation phenomenon of the radiation emitted by accretion disk around black holes may be quite common even with very simple physical ingredients of the models. Unfortunately this make harder to discern the difference between compact stellar object and black holes only on the basis of the time variability.

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