RELATIVISTIC STELLAR CLUSTERS

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ABSTRACT. Star clusters with very high densities may play an important role in QSO and nuclei of galaxies. This role is strongly influenced by relativistic instability which can be reached at different critical densities under particular conditions (e.g. the formation of massive black holes in AGN's). On the other hand it exists the possibility to have stable relativistic clusters with arbitrarily large central redshift (and density). The equilibrium and stability of relativistic clusters described by a Maxwellian distribution function with a cutoff in phase space is discussed. The results are compared and contrasted with ones existing in literature.

Key words: Stellar dynamics; general relativity.

1. Introduction

The question of existence of relativistic clusters is open since the discovery of quasars: are the clusters so dense that relativistic corrections to Newtonian theory modify their structure and influence their evolution? Theoretical calculations suggest that clusters might form in the nuclei of some galaxies and quasars, and the formation of massive black holes in quasars and active galactic nuclei (AGNs) could be a result of a collapse of dense stellar clusters. Nevertheless astronomical observations have yielded no definitive evidence about the existence of relativistic clusters, even if, with HST observations, there is the possibility to resolve this issue completely.

The study of models of equilibrium describing relativistic clusters and the investigation of the stability against relativistic collapse of such dense systems were developed in two different papers by Bisnovatyi-Kogan *et al.* (1993, 1998). In these papers were introduced three different stability methods similar to the static criteria for stars. These methods have been applied to sequences of equilibrium models, with different cutoff parameters in the distribution function, which generalize the ones studied by Zel'dovich & Podurets in 1965. Different regions of dynamical stability were discovered at different values of temperature and central redshift extending the range of stable configurations up to very large central densities.

The investigation of the models with different cutoff parameters arises from the necessity to consider all kind of cutoff realized in the stellar clusters. For giant elliptical galaxies, with small rotation and absence of a disk subsystem, we may expect very extended central objects, corresponding to large values of energy cutoff. For AGNs in spiral galaxies, like SyG, the cutoff is probably produced by the tidal action of the spirals and depends on the spiral structure in regions close to the center; here we may expect very small values of energy cutoff.

Dense stellar clusters are essentially represented by massive globular clusters with $M \sim 10^6 M_\odot$, active galactic nuclei and quasars with $M \sim 10^8 - 10^{10} M_\odot$, respectively.

2. History

Starting from the classical paper of Einstein (1939) on relativistic clusters, where a system formed by gravitating masses with circular motion around the center of symmetry was studied, the equilibrium and the dynamical stability of these systems was first analysed by Zel'dovich & Podurets (1965). In this paper was shown that the contraction of an high-density stellar cluster caused by evaporation of star may lead to a loss of stability and cause a relativistic collapse.

In the successive years this problem was developed in several works by Thorne (1966), Fackerell (1968) and Ipser (1969) who introduced virial methods in order to improve the results on stability. In particular the results by Ipser indicated a critical value of central redshift $z_c \sim 0.5$ for which instability occurs. On the other hand, in 1969, Bisnovatyi-Kogan & Zel'dovich had shown the existence of stable configurations with arbitrarily large central redshift for a particular set of solution with density distribution $\rho \sim \beta/r^2$.

The investigation of equilibrium and dynamical stability of relativistic clusters was extended by Suffern & Fackerell in 1976 and systematically studied and generalized in recent papers by Bisnovatyi-Kogan *et al.* (1993, 1998). In these papers the problem was defi-

nitely solved by the analysis of the behaviour of binding energy in the z_c-T diagram of the equilibrium solutions with the extension of the region of dynamically stable configurations even at values of central redshift larger than 0.5; in particular the main result was the existence of stable configurations with arbitrarily large central redshift as indicated by Bisnovatyi-Kogan & Zel'dovich in 1969, for a particular set of solutions, and by Merafina & Ruffini in 1995 with the introduction of an explicit relation between a particular family of stable solutions and the results of the numerical simulations obtained by Rasio *et al.* in 1989 indicating stable models with large values of z_c.

From thermodynamical point of view the results on the stability are incomplete. The methods used for stability analysis give results generally accepted only in Newtonian regime with the well known paper of Lynden-Bell & Wood in 1968 about gravothermal instability and the ones of Katz (1978, 1980), while in relativistic regime the problem is still open.

The problem of gravothermal catastrophe goes back to Antonov theorems (1962) and becomes very popular with the paper of Lynden-Bell & Wood (1968) in which the term "gravothermal catastrophe" was coined. Antonov's discovery was that no state of locally maximal entropy exists for stellar sistems of given energy $E < 0$ and mass M within a spherical box of radius greater than $R = 0.335 \, GM^2/(-E)$.

In Lynden-Bell and Wood paper the problem is analysed by studying the behaviour of a gaseous system confined in a spherical box with adiabatic walls. The gravitational equilibrium of the system is granted by the application of the virial theorem. Each particle of the system moves in the field generated by the other particles (mean field approximation). The result is that such a system has a negative total specific heat, with the core of the system with negative value and a *bath* surrounding the core with a positive value. Then, if we start with an isothermal equilibrium state and consider the effect of a perturbation which causes a flow of heat from an inner shell to an outer one, we have that this transfer from central regions will raise the temperature and the central density without limit inducing also the collapse of the core of the system. This phenomenon is the well known process called gravothermal catastrophe.

Clearly, gravothermal catastrophe is possible also in isothermal stellar systems where the equilibrium configurations are similar to gaseous spheres with equivalent velocity distribution. However, while in an isothermal gas gravothermal catastrophe develops by heat conduction with a timescale of the order of diffusion time (by collisions among the particles of the system), in a star cluster, due to stellar encounters, gravothermal catastrophe develops on a timescale of the order of the relaxation time. Globular clusters have a timelife larger than their relaxaton time and

therefore gravothermal catastrophe may be realistic in the evolution of these systems.

3. Spherical models

In order to analyze the stability of stellar systems we consider the equilibrium of an isothermal relativistic sphere of particles (stars), of the same mass, with a distribution function fulfilling Boltzmann statistics. The energy of the stars is limited by a cutoff in phase space and the distribution function is given by

$$
\begin{cases}
f = A \exp(-E/T) & \text{for } E \leq E_{cut} \\
f = 0 & \text{for } E > E_{cut},
\end{cases}
\tag{1}
$$

where $E_{cut} = mc^2 - \alpha T/2$ is the cutoff energy of the stars and T is the temperature "measured by an infinitely remote observer", in energy units, constant on each single equilibrium configuration. The parameter α is a constant for each configuration and can vary from 0 to 2.87. The upper limit on α is a condition on the existence of equilibrium solutions (see Bisnovatyi-Kogan *et al.*, 1998). For $\alpha = 1$ we recover the distribution considered by Zel'dovich & Podurets in 1965.

The equations of gravitational equilibrium for a spherically symmetric system described by the Schwarzschild metric $ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2 d\phi^2)$ are given by

$$
\begin{cases}
e^{-\lambda} \left(\frac{1}{r} \frac{d\nu}{dr} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^4} P & \text{with } \nu(0) = \nu_0 \\
e^{-\nu} \left(\frac{1}{r} \frac{d\lambda}{dr} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{8\pi G}{c^4} \varepsilon & \text{with } \lambda(0) = 0,
\end{cases}
\tag{2}
$$

where P is the pressure and $\varepsilon = \rho c^2$ is the energy density. The expressions of these thermodynamical quantities are easily obtained from the distribution function given in Eq. 1. We have

$$
P = \frac{4\pi A}{3c^3 e^{\nu/2}} \int_{mc^2 e^{\nu/2}}^{mc^2 - \alpha T/2} e^{-E/T} (e^{-\nu} E^2 - m^2 c^4)^{3/2} dE
\tag{3}
$$

and

$$
\varepsilon = \frac{4\pi A}{c^3 e^{3\nu/2}} \int_{mc^2 e^{\nu/2}}^{mc^2 - \alpha T/2} e^{-E/T} \sqrt{e^{-\nu} E^2 - m^2 c^4} \, E^2 dE.
\tag{4}
$$

We obtain different families of equilibrium solutions depending on three papameters: α, T and z_c (central gravitational redshift). If we consider the entire range of possible values of α, we can obtain equilibrium configurations characterized by estreme core-halo density profiles as well as more homogeneous configurations. The results of these integrations are summarized in Fig.1. In this diagram we plotted the central redshift z_c as a function of the temperature T. Along each sequence of equilibrium models α is constant and T varies until a maximum value and become constant for large
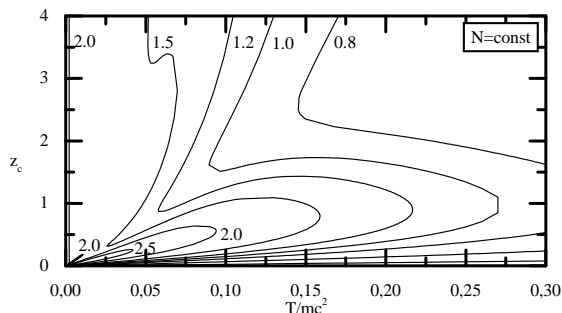
Figure 1: Sequences of equilibrium configurations with different values of α in the plane z_c-T.

z_c. For $\alpha = 1$ we obtain the solution of Zel'dovich & Podurets with the well known limit in the temperature $T/mc^2 = 0.227$.

The calculations have shown that equilibrium solutions exist only for values of $\alpha < 2.87$. It is also evident at small values of the temperature the α–curves deforme so that they become to consist of two separate branches: one looping in the origin ($z_c = 0$, $T = 0$) and one coming from. This fact is due to the behavior of the solutions in Newtonian regime where, for each value of α, more different equilibrium configurations may be.

4. Dynamical stability

Dynamical stability of isothermal configurations is studied since many years. Newtonian solutions are always stable against radial perturbations, being $df/dE < 0$ (Antonov, 1960). In relativistic regime the problem has been analysed by Ipser in 1969 and by Suffern & Fackerell in 1976 for configurations with sufficiently large values of the temperature T. The conclusions were that only configurations with redshift z_c smaller than 0.5 can be stable against radial perturbations. At low temperature regimes, the conclusions were uncertain, even if the possibility to have stable configurations with larger values of z_c was taken into account.

Now, for investigating the dynamical stability of dense stellar clusters with distribution function given by Eq. 1, we use three different approaches (Bisnovatyi-Kogan *et al.*, 1998).
1. Sequences of models with a fixed cutoff parameter, changing in accordance with the adiabatic condition $p_{cut} \sim n^{1/3}$.
2. Sequences of models with constant specific entropy.
3. Sequences of non-Maxwellian models, constructed from the condition of conservation of adiabatic invariant.

4.1. Sequences with a fixed cutoff parameter

The parameters

$$W_0 = \left(\frac{\epsilon_{cut}}{T_r} \right)_{r=0} \qquad \text{and} \qquad \beta = \frac{T_R}{mc^2} \,, \qquad (5)$$

where $T_r = Te^{-\nu/2}$ is the local temperature varying along the cluster and $\epsilon_{cut} = (p_{cut}^2 c^2 + m^2 c^4)^{1/2} - mc^2 = Ee^{-\nu/2} - mc^2$ the kinetic energy cutoff, are connected with T and α by the following relations

$$\beta = \frac{T}{mc^2 - \alpha T/2} \qquad (6)$$

and

$$W_0 = \frac{1 - e^{\nu(0)/2}}{T/mc^2} - \frac{\alpha}{2} \,. \qquad (7)$$

The first of the sequences used for stability analysis is the sequence with constant W_0 and varying β. The parameter W_0 can be taken as approximately adiabatic. The equivalence of this stability criterion with the one suggested by Ipser in 1980 was definitely shown by Bisnovatyi-Kogan *et al.* in 1993 for models with $\alpha = 1$: this sequence near the critical point corresponds to the one relevant in the application of the Ipser's criterion. This correlation near the critical point keeps also at $\alpha \neq 1$ for models with $T/mc^2 > 1.5$ and leads to results in accordance with the ones given in literature.

4.2. Sequences with a fixed specific entropy

By introducing the expression of the entropy of a system with arbitrary distribution function

$$S = \iint f(1 - \ln f) \, d^3p \, d^3r \,, \qquad (8)$$

we investigate the sequences with the fixed specific entropy S/N_0, where N_0 is the total number of stars. The expression of the specific entropy is

$$s \equiv \frac{S}{N_0} = [1 - \ln(A/A_*)] + \frac{\int_0^R e^{(\lambda+\nu)/2} \varepsilon r^2 dr}{T \int_0^R e^{\lambda/2} n r^2 dr} \,, \qquad (9)$$

where A_* is an arbitrary constant along the sequence with the dimension of A.

4.3. Sequence with the conservation of the adiabatic invariant

The conservation of the adiabatic invariant $I = pn_c^{-1/3}$ (see Podurets, 1969) along the sequence of models implies the introduction of non-Maxwellian distribution functions

$$f = A \exp \left\{ -\frac{e^{\nu/2}}{T_0} \left[p^2 c^2 \left(\frac{n_{c0}}{n_c} \right)^{2/3} + m^2 c^4 \right]^{1/2} \right\} \,, \qquad (10)$$

with the cutoff parameters

$$p_{cut} = p_{cut0} \left(\frac{n_c}{n_{c0}} \right)^{1/3} = \frac{p_{cut0}}{\kappa}, \quad \kappa = \left(\frac{n_{c0}}{n_c} \right)^{1/3} .$$
(11)

The expression for p_{cut0} is determined from the cutoff relation of the initial Maxwellian model with $T = T_0$ and $\nu = \nu_0(r)$. We have

$$(p_{cut0}^2 c^2 + m^2 c^4)^{1/2} \, e^{\nu_0(r)/2} = mc^2 - \alpha T/2 .$$
(12)

The procedure of construction of approximate equilibrium models with non-Maxwellian distribution function is described in details in the paper of Bisnovatyi-Kogan *et al.* (1993).

By using the three static criteria of dynamical stability mentioned above, we can construct a curve dividing the equilibrium solutions in two separate regions (see Fig.2 below). The results are in accordance for each different criterion. We have stable solutions in the region at small central redshift (close to T–axis) and in the region at small temperature (close to z_c–axis). This line seems to have an asymptotic behavior for large values of T or z_c: in the regime of small central redshifts, investigated by Ipser in 1969, there are no stable solutions with $z_c > 0.4832$ for large T; for small temperature there are no stable solutions with $T/mc^2 > 0.06$ for large z_c. Therefore it exists a new region of stable solutions which extends at even large z_c up to infinite central densities, with temperature T/mc^2 less than 0.06. This stable equilibrium configurations present a regular center without singularities even for very large density. However there is a very sharp separation between core and envelope, the core being up to only 10^{-4} times the radius of the cluster ! The core is in gravitational equilibrium with the external region: there is no possibility of existence for a so dense core without the envelope which permits to the system to be stable as a whole. Moreover the value of the ratio $2GM/Rc^2$ is small and of the order of the ones relevant for Newtonian configurations.

5. Thermodynamical stability

The role of thermodynamical instability in dense stellar clusters is not clear. The results in relativistic regime existing in literature are not yet definitive. Nevertheless it is possible to analyse the thermodynamical stability by applying the *linear series method*, first introduced by Poincaré in 1885, to the sequences which are relevant for this kind of perturbations. In fact, if we consider sequences of equilibrium configurations for which some quantities (invariant during the perturbations) are constant, then the first maximum

of binding energy in each sequence indicates the onset of instability. Sequences with $N = $ constant and $f[E_{cut}(r = R)] = $ constant, are relevant for thermodynamical stability.

This criterion was erroneously applied to these particular sequences by Ipser in 1980, in order to obtain the onset of dynamical instability. However, the result was correct anyway because both the onsets of thermodynamical and dynamical instability coincide in the region where the criterion was applied ($z_c \sim 0.5$ and $T/mc^2 > 0.06$). In fact, in that particular region of plane z_c-T, the maximum of binding energy is the same both for sequences with constant W_0, relevant in the analysis of dynamical stability, and for sequences relevant in the analysis of termodynamical instability. The equivalence of these two stability criteria was evidenced by Bisnovatyi-Kogan *et al.* (1993) and is easily deducible from the behaviour of binding energy as function of redshift and temperature (see 3-D diagram of Fig.8 in Bisnovatyi-Kogan *et al.* 1998) and from the sequences at constant $f[E_{cut}(r = R)]$ and W_0. Differences are indeed evident for configurations with sufficiently low temperatures ($T/mc^2 < 0.06$), where the critical curves bifurcate (see below).

In Newtonian regime we recover results in complete accordance with the ones given by Lynden-Bell and by Katz. In relativistic regime the results are shown in Fig.2, where the curve of the onset of thermodynamical instability is compared with the one relevant for dynamical instability in the z_c-T diagram.
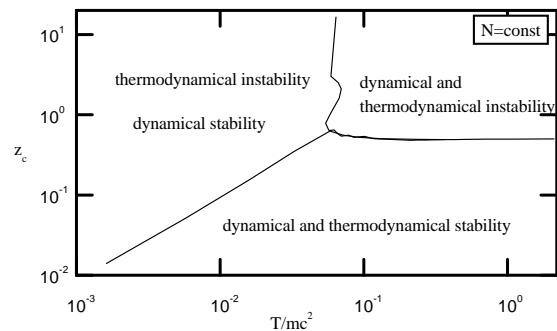

Figure 2: Dynamical and thermodynamical stability diagram in the plane N=const.

As preliminarily indicated, it is interesting to note that for $T/mc^2 > 0.06$ (large stars velocities) both curves coincide in correspondence to the well known critical value of central redshift $z_c \simeq 0.5$ obtained for dynamical instability. The two curves bifurcate for $T/mc^2 < 0.06$ (low stars velocities). The dynamical curve never reaches Newtonian regime and has an asymptotical behaviour towards a critical value of the temperature. All the configurations having a temperature lower than 0.06 mc^2 are dynamically stable for

arbitrarily large values of the central redshift. The thermodynamical curve, indeed, tends to Newtonian region and, for small values of the temperature coincides with the curve corresponding to the family of configurations with $W_0 = 7.6$ (in complete accordance with classical results). The parameter W_0 represents the gravitational potential expressed in terms of the local temperature (in energy units) at the center of the configuration. This parameter is connected with the main quantities by the relation

$$W_0 = \frac{mc^2}{T} \left(e^{\nu_R/2} - e^{\nu_0/2} \right). \qquad (13)$$

It is important to recall here some considerations about the evolution of a system which lost thermodynamical stability in consequence of collisions and evaporation of stars being still dynamically stable. Such a system begins to contract and heat his core, having central regions with negative specific heat (see Lynden-Bell & Wood 1968). This fact implies a motion in the z_c-T diagram towards the region of dynamical instability and the system will fatally cross the critical curve (vertical) at $T/mc^2 \simeq 0.06$, becoming dynamically unstable and leading to a faster collapse.

6. Conclusions

The results of the investigation on the stability against relativistic collapse of families of equilibrium configurations with different cutoff parameters can be summarized as follows (see Fig.2).

The region of the plane z_c-T with $T/mc^2 > 0.06$ corresponds to the traditional families of equilibrium configurations whose dynamical stability was largely investigated in the past. The three different criteria for investigating the stability give results in agreement among them and with the results of previous analysis (see, e.g., Ipser 1969). In this regime there are not stable configurations with z_c larger than 0.5. These results confirm this conclusion and are now obtained with more accuracy.

The region of the plane z_c-T with $T/mc^2 < 0.06$ corresponds to extreme core-halo configurations whose stability analysis carried out by Suffern & Fackerell (1976) did not supply conclusive results. In contrast with that conclusions the results show that these configurations are stable. The dense core is in gravitational equilibrium with a Newtonian envelope, which permits to the system to be stable as a whole. Thus we come to the interesting conclusion that *there exist stable non singular configurations with arbitrarily large central red-shift.*

It must be noted that the results of Ipser (1969) and Fackerell (1970), also reported by Suffern & Fackerell (1976), already indicated that models with small tem-

peratures could be stable even for values of the central red-shift larger than 0.5 but only until a limiting value of z_c. Nevertheless the authors came to a different conclusion by considering the behaviour of the curve of the maxima of the fractional binding energy. Application of these criteria to the particular solution obtained by Bisnovatyi-Kogan & Zel'dovich (1969) has shown that it satisfies the necessary condition for the stability, but is unable to establish the sufficient condition (Bisnovatyi-Kogan & Thorne 1970).

The results on the thermodynamical stability show that the critical curve of onset of thermodynamical instability lies at smaller values of central redshift than ones concerning the dynamical curve. From this follows that *thermodynamic stability always implies dynamical stability.*

Furthermore, while dynamical instability is reachable only in relativistic regime, thermodynamical instability can occur also in Newtonian regime. Consequently, thermodynamical instability always drives the system towards dynamical collapse which occurs only in relativistic regime after a contraction and heating of core. Therefore, even if in principle it is possible to have systems dynamically stable with an arbitrarily large central redshift, the core contraction induced by thermodynamical instability will lead anyway to a dynamical collapse.

References

Antonov V.A.: 1960, *AZh*, **37**, 918.

Antonov V.A.: 1962, *Vest Leningrad Univ*, **7**, 135.

Bisnovatyi-Kogan G.S., Zel'dovich Ya.B.: 1969, *Astrofizika*, **5**, 223.

Bisnovatyi-Kogan G.S., Thorne K.S.: 1970, *ApJ*, **160**, 875.

Bisnovatyi-Kogan G.S., Merafina M., Ruffini R., Vesperini E.: 1993, *ApJ*, **414**, 187.

Bisnovatyi-Kogan G.S., Merafina M., Ruffini R., Vesperini E.: 1998, *ApJ*, **500**, 217.

Einstein A.: 1939, *Ann. Math.*, **40**, 922.

Fackerell E.D.: 1968, *ApJ*, **153**, 643.

Ipser J.R.: 1969, *ApJ*, **158**, 17.

Ipser J.R.: 1980, *ApJ*, **238**, 1101.

Katz J.: 1978, *MNRAS*, **183**, 765.

Katz J.: 1980, *MNRAS*, **190**, 497.

Lynden-Bell D., Wood R.: 1968, *MNRAS*, **138**, 495.

Merafina M., Ruffini R.: 1995, *ApJ Lett*, **454**, L89.

Podurets M.A.: 1969, *AZh*, **46**, 126.

Poincaré H.: 1885, *Acta Math.*, **7**, 259.

Rasio F.A., Shapiro S.L., Teukolsky S.A.: 1989, *ApJ Lett*, **336**, L63.

Suffern K.G., Fackerell E.D.: 1976, *ApJ*, **203**, 477.

Thorne K.S.: 1966, *ApJ*, **144**, 201.

Zel'dovich Ya.B., Podurets M.A.: 1965, *AZh*, **42**, 963.