

THE NEGATIVE RESULT OF GRAVITATIONAL TESTS FOR KALUZA-KLEIN MODELS WITH SPHERICAL COMPACTIFICATION OF ADDITIONAL DIMENSIONS

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ABSTRACT. We investigated classical gravitational tests for the Kaluza-Klein model with spherical compactification of additional dimensions in the case of absence of a six-dimensional bare cosmological constant. We perturbed a background by a compact massive source with the dust-like equation of state in all spatial dimensions and obtained the solution of Einstein equations in the weak-field limit. It enabled to calculate PPN parameter γ , and we obtained a strong contradiction to observations.

Any theoretical model may be referred to physics only relative to the particular sphere of reality, where its findings are confirmed by experiment. Certainly, outside this sphere the theory represents just an abstractive logical construction and completely loses a right to be called a physical theory. Obviously, in such a context the Kaluza-Klein (KK) theory is not an exception and needs experimental verification.

There are a number of observable gravitational effects predicted by general relativity (GR). They include, as is well known, the Mercury perihelion shift, the deflection of light and the time delay of radar echoes (the Shapiro time-delay effect). In the weak-field approximation it's convenient to calculate all these effects using the so-called parameterized post-Newtonian (PPN) parameters β and γ [1]. These parameters are introduced as coefficients in the expansion of metrics in powers of a small parameter $2\varphi/c^2$ in a following way:

$$ds^2 \approx \left(1 + \frac{2\varphi}{c^2} + \beta \frac{2\varphi^2}{c^4} + \dots\right) c^2 dt^2 - \left(1 - \gamma \frac{2\varphi}{c^2} + \dots\right) \left(dr_3^2 + r_3^2 d\theta^2 + r_3^2 \sin^2 \theta d\phi^2\right) \quad (1)$$

Accordingly, experiments impose strict restrictions on these quantities. In particular, according to the Shapiro time-delay experiment using the Cassini spacecraft γ should be very close to the unity: $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$

[2]. This fact is in good agreement with GR, where $\gamma = 1$. Hence, separating the linear in $2\varphi/c^2$ mode similar to (1) in the certain multidimensional model, we can detect the deviation of theoretical predictions from experimental data. It's clear that the significant difference between γ and the unity points to the flaw in the considered theory.

Now let us proceed directly to the KK-theory analysis. Let's consider a factorizable 6-dimensional static background metrics

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - a^2 (d\xi^2 + \sin^2 \xi d\eta^2), \quad a \equiv \text{const}, \quad \eta \in [0, 2\pi), \quad \xi \in [0, \pi]. \quad (2)$$

It is defined on a product of the flat 4-dimensional (external) space-time and the 2-dimensional (internal) sphere with the radius a (in other words, a is the scale factor of the internal compact manifold). The metrics has topology $\mathbf{R} \times \mathbf{R}^3 \times S^2$. We choose the space with nonzero curvature intentionally. The case of the flat metrics with topology $\mathbf{R} \times \mathbf{R}^3 \times T^{D-3}$, where T^{D-3} is a $(D-3)$ -dimensional torus, has been investigated in [3]. As a result it was shown that in such a case $\gamma = 1/(D-2)$, and hence the condition $\gamma = 1$ is satisfied only in GR, where $D = 3$. The question is how common is this negative result for the Kaluza-Klein models. To understand it, we generalize the problem to the case of the curved metrics. In contrast to the models with toroidal compactification, in the present problem we need some bare matter to provide nonzero curvature of the internal space. Now we want to define the energy-momentum tensor (EMT) of this matter, using Einstein equations (also we consider the case of absence of a multidimensional cosmological constant):

$$\kappa T_{ik} = R_{ik} - \frac{1}{2} R g_{ik}, \quad \text{where } \kappa = 2S_5 \tilde{G}_6 / c^4. \quad (3)$$

Here $S_5 = 8\pi^2/3$ is a total solid angle and \tilde{G}_6 is a gravitational constant in the 6-dimensional space-time. It is obvious that the only contribution to the scalar curvature is provided by the Ricci tensor components corre-

sponding to the internal compact manifold. Using general formulae, we easily compute those components and the curvature (for details see [4]):

$$R_{44} = 1, R_{55} = \sin^2 \xi,$$

the trace of the Ricci tensor is

$$R = R_i^i = R_{44}g^{44} + R_{55}g^{55} = -\frac{2}{a^2}. \quad (4)$$

Substitution of (4) into (3) gives us a desired EMT of the background matter. It has the following form:

$$T_{ik} = \begin{cases} \frac{1}{\kappa a^2} g_{ik}, & \text{for } i = k = 0, \dots, 3; \\ 0, & \text{for } i = k = 4, 5; \end{cases} \quad \text{it's convenient to} \\ \text{introduce } \Lambda_4 \equiv \frac{1}{\kappa a^2}. \quad (5)$$

Clearly, such matter can be simulated by a perfect fluid with the vacuum equation of state in the external space and the dust-like equation of state in the internal one.

So, we found out the form of matter that corresponds to the considered geometric background, and now we intend to perturb this background by a static point-like mass. It's well known that a point-like mass is a good approximation to calculate classical gravitational tests in GR. And certainly, the physical content of this approximation preserves under the transition to multidimensionality. Thus, we make a perturbation of the background by a static massive source insertion. Let the perturbation have its non-relativistic rest mass density of the form $\varepsilon\rho(\mathbf{r}_5)$. Here ε is an infinitesimal prefactor introduced to simplify keeping of the perturbation orders during calculations and $\rho(\mathbf{r}_5)$ is a certain function of all spatial coordinates. There are two separate cases of this function's form. In the first case the source with the rest mass m is uniformly smeared over the internal sphere and has its multidimensional density of the form $\rho = \rho_3(\mathbf{r}_3)/(4\pi a^2) = m\delta(\mathbf{r}_3)/(4\pi a^2)$. In the second case a particle is localized on the sphere: $\rho = \rho(\mathbf{r}_5) = m\delta(\mathbf{r}_5)$. Indeed, the problem with delocalized perturbation is of more full physical sense, but we shall investigate the case without smearing, that is more general from the mathematical standpoint. We present the perturbed metrics in the following form:

$$ds^2 = A(\mathbf{r}_5)c^2 dt^2 + B(\mathbf{r}_5)dx^2 + C(\mathbf{r}_5)dy^2 + D(\mathbf{r}_5)dz^2 + E(\mathbf{r}_5)d\xi^2 + F(\mathbf{r}_5)d\eta^2. \quad (6)$$

Up to corrections of the first order in perturbation all unknown functions may be rewritten in such a way:

$$A \approx A^0 + \varepsilon A^1(\mathbf{r}_5), \quad B \approx B^0 + \varepsilon B^1(\mathbf{r}_5), \quad C \approx C^0 + \varepsilon C^1(\mathbf{r}_5), \\ D \approx D^0 + \varepsilon D^1(\mathbf{r}_5), \quad E \approx E^0 + \varepsilon E^1(\mathbf{r}_5), \quad F \approx F^0 + \varepsilon F^1(\mathbf{r}_5). \quad (7)$$

The terms indexed by zero correspond to the background metric components. Also we suppose that the diagonal form of the metric tensor is preserved, and we show below that Einstein equations have a solution in a suchlike assumption. Really, let's try to solve the field equations rewritten as follows:

$$R_{ik} = \kappa \left(T_{ik} - \frac{1}{4} T g_{ik} \right), \quad (8)$$

where T_{ik} is the total EMT.

We can present the total EMT as a superposition $T_{ik} = \hat{T}_{ik} + \tilde{T}_{ik}$, where the first term is the EMT of the perturbation with the only component in the nonrelativistic limit $\hat{T}_{00} \approx \varepsilon\rho(\mathbf{r}_5)c^2$ (up to infinitesimals of the higher order). The second term is the EMT of the background. We suppose that the perturbation existence results in appearance of a small fluctuation $\Lambda_4 \rightarrow \Lambda_4 + \varepsilon\Lambda_4^{(1)}$ in (5). Therefore, it's not difficult to obtain explicit expressions for nonzero components of T_{ik} :

$$T_{00} \approx \frac{1}{\kappa a^2} + \varepsilon \left(\frac{1}{\kappa a^2} A^1 + \rho c^2 + \Lambda_4^{(1)} \right), \\ T_{ii} \approx -\frac{1}{\kappa a^2} + \varepsilon \left(\frac{1}{\kappa a^2} B_i - \Lambda_4^{(1)} \right), \\ B_1 = B^1, B_2 = C^1, B_3 = D^1. \quad (9)$$

Using bulky formulae, we can find expressions for the linearized in ε Ricci tensor components (this procedure is performed in more detail in [4]). Analysis of the obtained expressions along with non-diagonal Einstein equations is very helpful. It enables to reduce all field equations (8) to one equation and five conditions, presented below:

$$\Delta_3 A^1 + \frac{1}{a^2} \Delta_{\xi\eta} A^1 = \frac{3}{2} \kappa \rho c^2, \\ B^1 = C^1 = D^1 = E^1/a^2 = A^1/3, \quad \kappa \Lambda_4^{(1)} = \frac{E^1}{a^4}. \quad (10)$$

In our notation $\Delta_{\xi\eta}$ is the Laplace operator on the internal sphere. Obviously, the introduction of $\Lambda_4^{(1)}$ is justified, because only in the case $\Lambda_4^{(1)} \neq 0$ the internal manifold is compact ($a < +\infty$). Hence, making a change of the function $A^1 \equiv 2\varphi/c^2$ we come to the Poisson equation for the gravitational potential of the particle:

$$\Delta_3 \varphi + \frac{1}{a^2} \Delta_{\xi\eta} \varphi = S_5 G_6 \rho(\mathbf{r}_5), \quad \text{where } G_6 = 3\tilde{G}_6/2, \quad (11)$$

which admits the following solution:

$$\varphi = -\frac{S_5 G_6 m}{4\pi a^2 r_3} \sum_{l=0}^{+\infty} \sum_{m=-l}^l Y_{lm}^*(\xi_0, \eta_0) Y_{lm}(\xi, \eta) \exp\left(-\frac{\sqrt{l(l+1)}}{a} r_3\right). \quad (12)$$

Here ξ_0, η_0 denote the position of the source on a two-sphere and Y_{lm} are Laplace's spherical harmonics. It is evident that at large r_3 the obtained potential should coincide with the Newtonian one. From this boundary condition we easily get the following correlation: $S_5 G_6 / (4\pi a^2) = 4\pi G_N$, where G_N is the Newtonian gravitational constant. Therefore, the perturbation of the 00 metric coefficient reads

$$A^1 = -4\pi \frac{r_g}{r_3} \sum_{l=0}^{+\infty} \sum_{m=-l}^l Y_{lm}^*(\xi_0, \eta_0) Y_{lm}(\xi, \eta) \exp\left(-\frac{\sqrt{l(l+1)}}{a} r_3\right), \\ r_g = 2G_N m/c^2 \quad \text{is the gravitational radius.} \quad (13)$$

All the rest of unknown metric coefficients expresses via $A^1 \equiv 2\varphi/c^2$ from the conditions in (10).

Now we have requisite to find the PPN-parameter γ . Obviously, the characteristic sizes of astrophysical objects, such as the Sun, are much larger than the compactification scale of the internal space ($R_3 \gg a$). Then for

$r_3 \gg R_3$ we can limit ourselves to the zero mode in (13). Hence, at these distances the metrics (6) reads

$$ds^2 \approx \left(1 - \frac{r_g}{r_3}\right) c^2 dt^2 - \left(1 + \frac{1}{3} \frac{r_g}{r_3}\right) (dx^2 + dy^2 + dz^2) - a^2 \left(1 + \frac{1}{3} \frac{r_g}{r_3}\right) (d\xi^2 + \sin^2 \xi d\eta^2) \quad (14)$$

It can be easily seen from (14) that the PPN-parameter $\gamma = 1/3$. It's worth to note that the case of the uniformly smeared particle over the two-sphere is a consequence from the obtained result, and there is no effect on γ . We see that the obtained result is in complete contradiction to the observational data, because to satisfy the experimental constraints this quantity should be very close to the unity. We also note that the relation $\gamma = 1/(D-2)$ is right (in the present case $D=5$). All these facts indicate the presence of significant physical flaw in the considered models. We suggested that the problem is that in both of these types of

models (i.e. with toroidal and spherical compactification) the internal spaces are not stabilized. In [5] we show that our guess is correct and in the case of stabilized internal spaces considered models can be in agreement with observations.

References

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