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QUASI-ANALYTICAL METHOD FOR IMAGES CONSTRUCTION FROM GRAVITATIONAL LENSES

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ABSTRACT. One of the main problems in the study of system of equations of the gravitational lens, is the computation of coordinates from the known position of the source. In the process of computing finds the solution of equations with two unknowns. The problem is that, in general, there is no analytical method that can find all of the roots (lens) of system over the field of real numbers. In this connection, use numerical methods like the method of tracing. For the N-point gravitational lenses we have a system of polynomial equations. The methods of algebraic geometry, we transform the system to another system, which splits into two equations. Each equation of the transformed system is a polynomial in one variable. Finding the roots of these equations is the standard computing task.

Keywords: Lens: gravitational lenses, binary lenses; Algebraic geometry: resultant.

1. Introduction

According to the general theory of relativity, the light beam, which passes close to a point source of gravity (gravitational lens) at a distance ξ from it (in case $\xi \gg r_g$) is deflected by an angle

$$\vec{\alpha} = \frac{2r_g}{\xi^2}\vec{\xi} = \frac{4GM}{c^2\xi^2}\vec{\xi}, \tag{1}$$

where r_g – gravitational radius; M – mass point of the lens; G – gravity constant; c – velocity of light in vacuum. The detailed derivation of the formula (1) can be found in many classic books (Blioch et al., 1989; Weinberg, 1972; Landau et al., 1988). For N-point of the gravitational lens, in the case of small tilt angles have the following equation (Zakharov, 1997; Schneider et al., 1999) in dimensionless variables:

$$\vec{y} = \vec{x} - \sum_i m_i \frac{\vec{x} - \vec{l_i}}{\left|\vec{x} - \vec{l_i}\right|^2}, \tag{2}$$

where $\vec{l_i}$ – dimensionless radius vector of point masses outside the lens, and the mass m_i satisfy the relation $\sum m_i = 1$.

The Equation (2) in coordinate form has the form of system:

$$\begin{cases} y_1 = x_1 - \sum\limits_{i=1}^{N} m_i \frac{x_1 - a_i}{(x_1-a_i)^2+(x_2-b_i)^2} \\ y_2 = x_2 - \sum\limits_{i=1}^{N} m_i \frac{x_2 - b_i}{(x_1-a_i)^2+(x_2-b_i)^2} \end{cases}, \tag{3}$$

where a_i and b_i are the coordinates of the radius-vector $\vec{l_i}$, i.e. $\vec{l_i} = (a_i, b_i)$.

2. Mathematical definitions and theorems

For conversion of system (3) we needs the following definitions and theorems.

Definition 1. Let K – an arbitrary field of numbers, and $f(x), g(x)$ – polynomials of $K[x]$ over this field. The resultant $R(f,g)$ of the polynomials $f(x), g(x)$ is called an element of K, which we have defined by the formula:

$$R(f,g) \equiv a_0^m b_0^n \prod\nolimits_{i=0}^{i=n} \prod\nolimits_{j=0}^{j=m} (\alpha_i - \beta_i), \tag{4}$$

where α_i, β_i- the roots of polynomials, $f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$ and $g(x) = \sum_{j=0}^{j=m} b_j x^{m-j}$, accordingly, with the leading coefficients of the polynomials, such that $a_0 \neq 0$, $b_0 \neq 0$.

Defenition 2. Matrix Sylvester for polynomials $f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$ and $g(x) = \sum_{j=0}^{j=m} b_j x^{m-j}$ is called a square matrix $S = S(f,g)$ of the order of $n+m$ elements of which are defined by s_{ij},

$$S = [s_{ij}] = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & \dots & & \\ & a_0 & a_1 & a_2 & \dots & & \\ & & \dots & \dots & \dots & & \\ & & & & & a_{n-1} & a_n \\ b_0 & b_1 & b_2 & \dots & \dots & & \\ & b_0 & b_1 & b_2 & \dots & & \\ & & \dots & \dots & \dots & & \\ & & & & & b_{m-1} & b_m \end{bmatrix}, \tag{5}$$

where the number of rows with coefficients a_i is equal to $m = \deg g(x)$, and the number of rows with equal coefficients b_j is equal to $n = \deg f(x)$.

The resultant and Sylvester matrix connects the following

Theorem 1. Let $R(f, g)$ – the resultant of f and g, and $S(f, g)$ - they Sylvester matrix, then:

$$R(f, g) = \det S(f, g) \tag{6}$$

The proof of Theorem 1, see. eg, (Lang 1965, Van Der Waerden 1971).

We have the following

Theorem 2. The polynomials f and g have a common root, if and only if,

$$R(f, g) = 0. \tag{7}$$

The modern proof of Theorem 2 sees. Eg., (Lang, 1965; Van Der Waerden, 1971).

From Theorem 2 follow

Theorem 3. Let $A = A(x_1, x_2)$ and $B = B(x_1, x_2)$ are polynomials of two variables x_1 and x_2. System of equations

$$\begin{cases} A(x_1, x_2) = 0 \\ B(x_1, x_2) = 0 \end{cases}, \tag{8}$$

has a solution, if and only if, at least one of the results $R_{x_1}(A, B)$, $R_{x_2}(A, B)$ is zero.

The proof of Theorem 3, see. Eg (Walker, 1950; Van Der Waerden, 1971).

There is also

Theorem 4. The polynomials $A = A(x_1, x_2)$ and $B = B(x_1, x_2)$ have a common component, if and only if, at least one of the results, $R_{x_1}(A, B)$, $R_{x_2}(A, B)$ is identically equal to zero.

3. The algorithm for constructing images for N-point lens

Theorems 1-3 we use the algorithm for solving the problem of calculating the coordinates of the image points. The right parts the equations of system (3), are rational functions of the variables x_1 and x_2. We transform each equation of the system (3) in a polynomial equation, and we obtain a system of equations:

$$\begin{cases} F_1(x_1, x_2, y_1) = 0 \\ F_2(x_1, x_2, y_2) = 0 \end{cases}. \tag{9}$$

We apply to the system (9) Theorem 4. Let's see has the common component the equations of system. If there is a common component, we separate it. In this case, the system (9) is divided into several sub-systems with the equations without common components. If there is no common components, we are using theorem 3, and we are eliminate from system (9) of the variable x_1, and then the variable x_2. The two obtained equations form a system

$$\begin{cases} R_1(x_2, y_1, y_2) = 0 \\ R_2(x_1, y_1, y_2) = 0 \end{cases}. \tag{10}$$

System (10) from the system (9), but in general it is not equivalent. The first equation of the system is a polynomial in the variable x_2. The second – a polynomial in the variable x_1. The variables y_1 and y_2 we believe parameters. We compute the set of roots of each of R_1 and R_2 polynomials and select them to a subset of real roots. Determine the direct multiplication of the sets of real roots of polynomials. We check each pair of direct multiplication so that it satisfies the system (9). This selection is completely defines the set of solutions of this system. Thus, in the first stage, a solution of (3) we are reduced to the computation of roots of polynomials in one variable. We achieve this by using analytical methods of classical algebraic geometry. In the second stage, we apply approximate methods of calculation. The problem is reduced to the computation of roots of polynomials in one variable over the field of complex numbers.

4. The binary lens

The binary lens (in the case of $N = 2$) were studied in many works, see. Eg, (Schneider et al., 1999, Schneider et al., 1986, Asada 2002, Cassan 2008, Witt 1990).To demonstrate the method, we consider the two-point gravitational lens with equal masses $m_1 = m_2 = 1/2$. Point masses are on the x – axis at a distance a from the origin to (Figure 1).

$$\begin{cases} y_1 = x_1 - \frac{1}{2}\frac{x_1 - a}{(x_1-a)^2 + x_2^2} - \frac{1}{2}\frac{x_1 + a}{(x_1+a)^2 + x_2^2} \\ y_2 = x_2 - \frac{1}{2}\frac{x_2}{(x_1-a)^2 + x_2^2} - \frac{1}{2}\frac{x_2}{(x_1+a)^2 + x_2^2} \end{cases}. \tag{11}$$

In this case, the system (3) takes the form:We transform the system (11) to (9) and we represent the polynomials F_1 and F_2 in the lexicographical form in ascending powers of the variable x_1. We have:

$$\begin{cases} F_1 = -y_1(a^4 + 2a^2x_2^2 + x_2^4) + (a^2 - x_2^2 + \\ \quad +(a^2 + x_2^2)^2)x_1 + 2y_1(a^2 - x_2^2)x_1^2 + \\ \quad +(2x_2^2 - 1 - 2a^2)x_1^3 - y_1x_1^4 + x_1^5, \\ F_2 = (-a^2x_2 - x_2^3 + (a^2 + x_2^2)^2(x_2 - y_2)) - \\ \quad -(x_2 + 2(a^2 - x_2^2)(x_2 - y_2))x_1^2 + \\ \quad +(x_2 - y_2)x_1^4. \end{cases} \tag{12}$$

With the resultant $R_1 = R(F_1, F_2)$ exclude from the system (12) variable x_1. Sylvester matrix $S_1 = S(F_1, F_2)$ is of the order $\deg F_1 + \deg F_2 = 9$. Since

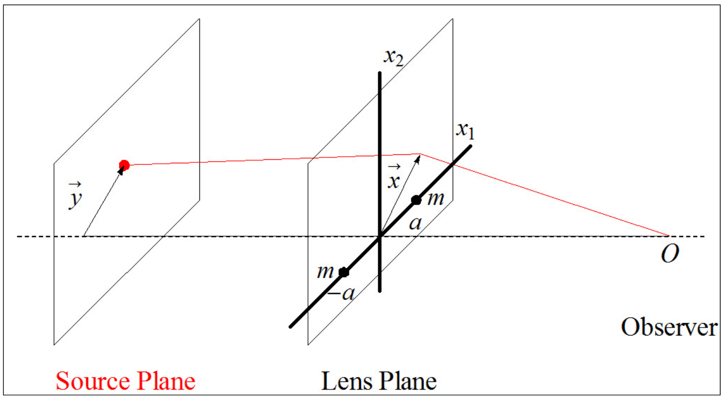

Figure 1: The binary gravitational lens.

$R_1 = \det S_1$, we have:

$$\begin{gathered}R_1(x_2,y_1,y_2) = 4a^4x_2^2(a^2+x_2^2)(-a^2y_2^3-\\ -y_2^2(1-a^2+4a^4+4a^2y_1^2+4a^2y_2^2)x_2-\\ -y_2(4a^2-4a^4+4a^6+y_1^2+4a^2y_1^2-\\ -8a^4y_1^2+4a^2y_1^4+5y_2^2-4a^2y_2^2+\\ +8a^4y_2^2+8a^2y_1^2y_2^2+4a^2y_2^4)x_2^2+\\ +(4a^6-4a^4+y_1^2+4a^2y_1^2-8a^4y_1^2+\\ +4a^2y_1^4+y_2^2-12a^2y_2^2+8a^4y_2^2-8y_1^2y_2^2+\\ +8a^2y_1^2y_2^2-8y_2^4+4a^2y_2^4)x_2^3-\\ -4y_2(a^4-a^2-y_1^2-2a^2y_1^2+y_1^4-\\ -y_2^2+2a^2y_2^2+2y_1^2y_2^2+y_2^4)x_2^4+\\ +4(a^4+2a^2(y_2^2-y_1^2)+(y_1^2+y_2^2)^2)x_2^5).\end{gathered} \tag{13}$$

Similarly calculate the resultant R_2:

$$\begin{gathered}R_2(x_1,y_1,y_2) = 4a^4x_1^2(a^2-x_1^2)(-a^2y_1^3+\\ +y_1^2(1+a^2+4a^4-4a^2y_1^2-4a^2y_2^2)x_1-\\ -y_1(4a^2+4a^4+4a^6-5y_1^2-4a^2y_1^2-\\ -8a^4y_1^2+4a^2y_1^4-y_2^2+4a^2y_2^2+\\ +8a^4y_2^2+8a^2y_1^2y_2^2+4a^2y_2^4)x_1^2+\\ +(4a^4+4a^6-y_1^2-12a^2y_1^2-8a^4y_1^2+\\ +8y_1^4+4a^2y_1^4-y_2^2+4a^2y_2^2+8a^4y_2^2+\\ +8y_1^2y_2^2+8a^2y_1^2y_2^2+4a^2y_2^4)x_1^3+\\ +4y_1(a^2+a^4-y_1^2-2a^2y_1^2+y_1^4-\\ -y_2^2+2a^2y_2^2+2y_1^2y_2^2+y_2^4)x_1^4-\\ -4(a^4-2a^2y_1^2+y_1^4+2a^2y_2^2+2y_1^2y_2^2+y_2^4)x_1^5).\end{gathered} \tag{14}$$

If there are the coordinates y_1 and y_2 of source, coordinates of images x_1 and x_2 can be calculated in several ways.

The first method. Numerically solve the equations $R_1(y_1,y_2,x_2) = 0$ and $R_2(y_1,y_2,x_1) = 0$. Determine the sets of their roots $x_1^{(i)}\{x_1^{(1)},x_1^{(2)}...x_1^{(p)}\}$ and $x_2^{(j)}\{x_2^{(1)},x_2^{(2)}...x_2^{(q)}\}$, respectively. Select a subsets of real roots. Determine their direct multiplication of. Couples $(x_1^{(i)},x_2^{(j)})$ – elements of direct multiplication, substitute into the original system of equations (11). Choose solutions that satisfy the system. It uniquely identifies the set of solutions of the system (11).

The second method. Numerically solve one of the equations (14), such as the first. Calculate the set of its roots $x_2^{(j)}\{x_2^{(1)},x_2^{(2)}...x_2^{(q)}\}$. Select a subset of real roots. Each of the real roots substitute into the equation of system (11). Find the solutions $x_1^{(i)}\{x_1^{(1)},x_1^{(2)}...x_1^{(p)}\}$ that correspond to each of the roots of x_2.

5. Conclusions

In this article we offer a quasi-analytic method of solution of the vector equation N-point gravitational lens. The method consists of two main stages. Analytical stage, in which we reduce the problem to the solution of systems of polynomial equations, which, in turn, reduce to the problem of finding the roots of polynomials in one variable. The numerical stage, in which we calculate the roots of polynomials in one variable. A complete solution we get when we separate real solutions and we tested them. Thus, our method is both more accurate and faster.

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