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TOWARD THE QUANTIZATION OF BLACK HOLES

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ABSTRACT. In order to construct a quantum model of black hole (BH), we introduce a modified description of classical space-time BH (the Schwarzschild solution). We develop the Lagrangian formalism of the vacuum gravitational field in spherically symmetric space-time, divided on the two regions: R- and T-regions. Initial metrics in their regions are taken in the scale-invariant form and depend on a timelike coordinate in T-region and space-like coordinate in R-region. We introduce the Hamiltonian and mass function, which corresponding evolutional coordinate (t or r) in each of regions. Their Poisson brackets are proportional Hamiltonian constraint. Further we construct the quantum operators of Hamilton and masse. Their commutator are proportional to the Hamilton operator. System of Wheeler-De Witt equation and equation on the own values of mass operator, together with the compatibility condition, allow to find wave functions in every region. These wave functions form the common wave function of BH with the continuous masse spectrum.

Keywords: Black hole, mass function, Hamiltonian constraint, quantum, mass operator, compatibility condition.

1. Introduction

Beginning with article Bekenstein (1974) it is considered that BH has a discrete mass spectrum. This connect with the idea of quantizing the horizon area of BH. The last was based on the fact that the horizon area of a nonextremal BH behaves in a sense as an adiabatic invariant (Christodoulou, 1970,1971). In what follows, these ideas developed in numerous works (See e.g., Mukhanov, 1986; Bekenstein et all, 1995; Barvinsky et al, 1996, 2001; Hod, 1998; Khriplovich 1998, 2002, 2004, 2008, etc). On the other hand, the known approaches (Thiemann at all, 1993, Kastrup at all, 1994, Kuchař, 1994) to the canonical quantization of the spherically-symmetric gravitational field give the continuous mass spectrum. Formal canonical approach (Cavaglia et al, 1995,1996) gives an analogical result. These approaches can be related by postulating periodic boundary conditions in time for the plane waves and by identifying the period Δ in real time with the period $\Delta_H = 8\pi G M/c^3$ in Euclidean time (Kastrup, 1996). This yields the mass spectrum $M_n = (1/2)m_P\sqrt{n}\,, n = 1, 2, \cdots$. In the work Jalalzadeh et al (2011) by using the modification of gauge multiplier and with the application of Wheeler-DeWitt approach the discrete mass spectrum of BH was also got. In this work we, on the basis of simple geometrodynamical approach with the use of DeWitt equation and quantum mass operator, we construct quantum model of BH with the continuous mass spectrum.

2. Classic space-time description

2.1. Actions, Lagrangians and supermetrics in the R- and T-regions

Spherically-symmetric space-time (ST) with the metric

$$ds^2 = h\,(r,t)\left(dx^0\right)^2 - g\,(r,t)\,dr^2 - R^2\,(r,t)\,d\sigma^2\,, \quad (1)$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta d\alpha^2$, has the scalar curvature

$$\begin{aligned} {}^{(4)}R &= 2\left\{-1 - \frac{R}{g}R_{,0}\left(\ln\left(hR\right)\right)_{,0} + \right. \qquad (2)\\ &+ \left. \frac{R}{h}R_{,1}\left(\ln(gR)\right)_{,1}\right\}\sin\theta + \left(div\vec{V}\right)\sin\theta\,. \end{aligned}$$

Here $R_{,0} = \partial R/\partial x^0$, $R_{,1} = \partial R/\partial r$, $div\vec{V}$ is divergence some vector $\vec{V}$. Action for a free gravitational field

$$S = -\frac{c^3}{16\pi\kappa}\int_{V^4}\sqrt{-g}^{(4)}Rd^4x\,, \quad (3)$$

after reduction and rejection of the surface term, can be written as

$$S = \int_{V^2}\tilde{L}dx^0dr\,, \quad (4)$$

where $\tilde{L}$ – effective Lagrangian

$$\tilde{L} = \frac{c^3R}{2\kappa}\left\{\sqrt{\frac{h}{g}}R_{,r}\left(\ln\left(hR\right)\right)_{,r} - \sqrt{\frac{g}{h}}R_{,0}\left(\ln(gR)\right)_{,0} + 1\right\}. \quad (5)$$

Information about the ST structure is contained in the term $(\nabla R)^2 = \gamma^{ab} R_{,a} R_{,b}$. The surface $R(r, x^0) = R_g = const$, for which the equality $(\nabla R)^2 = 0$ holds, divides ST on two parts: T- and R-regions. Besides, in T: $(\nabla R)^2 > 0$, and in R: $(\nabla R)^2 < 0$. Taking into account the Birkhoff theorem, in the R- or T-region can be chosen such coordinate systems in which metrical coefficients depend only on space-like r or timelike x^0 coordinate, so that:

$$ds_+^2 = h_+(r)\left(dx^0\right)^2 - g_+(r)\, dr^2 - R_+^2(r)\, d\sigma^2\,, \qquad (6)$$

$$ds_-^2 = h_-\left(x^0\right)\left(dx^0\right)^2 - g_-\left(x^0\right) dr^2 - R_-^2\left(x^0\right) d\sigma^2\,. \quad (7)$$

We will define the dimensionless variables $\zeta_\pm,\ \tau_\pm$:

$$r = \frac{R_g}{2}\zeta_\pm,\ x^0 = \frac{R_g}{2}\tau_\pm,\ r_g = \frac{R_g}{2}\zeta_g,\ x_g^0 = \frac{R_g}{2}\tau_g, \ (8)$$

where $\{\zeta_-, \tau_-\} \in T$, $\{\zeta_+, \tau_+\} \in R$. Then

$$ds_+^2 = h_+(\zeta_+)(d\tau_-)^2 - g_+(\zeta_+)(d\zeta_+)^2 - R_+^2(\zeta_+) d\sigma^2\,,$$
$$ds_-^2 = h_-(\tau_-)(d\tau_-)^2 - g_-(\tau_-)(d\zeta_-)^2 - R_-^2(\tau_-) d\sigma^2\,.$$

By virtue of additivity, the action can be rewritten as follows

$$S = S_- + S_+ = \int\limits_0^{x_g} L_- d\tau_- + \int\limits_{x_g}^{\infty} L_+ d\zeta_+\,, \qquad (9)$$

$$L_- = \frac{s_0}{2}\sqrt{h_- g_-}\left\{-\frac{R_-}{h_-} R_{-,0}\left(\ln\left(g_- R_-\right)\right)_{,0} + 1\right\}, \ (10)$$

$$L_+ = \frac{s_0}{2}\sqrt{h_+ g_+}\left\{\frac{R_+}{g_+} R_{+,1}\left(\ln\left(h_+ R_+\right)\right)_{,1} + 1\right\}, \ (11)$$

where $s_0 = R_g^2 c^3/4\kappa$. In (9) integration on ζ_- and τ_+ is executed in the interval $l = \zeta_{-2} - \zeta_{-1} = \tau_{+2} - \tau_{+1} = 1$. We will enter the new field variables:

$$h_- = \frac{n_- + u_-}{n_- - u_-} N_-^2, g_- = \frac{n_- - u_-}{n_- + u_-}, R_- = \frac{R_g}{2}\left(n_- + u_-\right), \ (12)$$

$$h_+ = \frac{u_+ - n_+}{u_+ + n_+}, g_+ = \frac{u_+ + n_+}{u_+ - n_+} N_+^2, R_+ = \frac{R_g}{2}\left(u_+ + n_+\right). \ (13)$$

Then metrics and Lagrangians take the form of

$$ds_-^2 = \frac{R_g^2}{4}\left\{N_-^2 \frac{n_- + u_-}{n_- - u_-} d\tau_-^2 - \frac{n_- - u_-}{n_- + u_-} d\zeta_-^2 - \right.$$
$$\left. - \left(n_- + u_-\right)^2 d\sigma^2\right\}, \quad |u_-| < n_-, \qquad (14)$$

$$L_- = \frac{s_0}{2}\left\{\frac{1}{N_-}\left(u_{-,\tau_-}^2 - n_{-,\tau_-}^2\right) + N_-\right\}, \qquad (15)$$

$$ds_+^2 = \frac{R_g^2}{4}\left\{\frac{u_+ - n_+}{u_+ + n_+} d\tau_+^2 - N_+^2 \frac{u_+ + n_+}{u_+ - n_+} d\zeta_+^2 - \right.$$
$$\left. - \left(u_+ + n_+\right)^2 d\sigma^2\right\}, \quad 0 < n_+ < u_+ < \infty\,, \qquad (16)$$

$$L_+ = \frac{s_0}{2}\left\{\frac{1}{N_+}\left(u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2\right) + N_+\right\}, \qquad (17)$$

where $_{,\tau_\pm} = \partial/\partial\tau_\pm$, $_{,\zeta_\pm} = \partial/\partial\zeta_\pm$. The requirement $\delta S = 0$ with respect variations δN leads to constrains

$$\frac{\partial L_-}{\partial N_-} = 0 \Longrightarrow u_{-,\tau_-}^2 - n_{-,\tau_-}^2 = N_-^2\,, \qquad (18)$$

$$\frac{\partial L_+}{\partial N_+} = 0 \Longrightarrow u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2 = N_+^2\,. \qquad (19)$$

Except the Lagrange multipliers N from L, we get Lagrangians and actions of the system in both minisuperspaces:

$$L_- = s_0\sqrt{u_{-,\tau_-}^2 - n_{-,\tau_-}^2},\ L_+ = s_0\sqrt{u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2}, \ (20)$$

$$S_- = s_0\int\limits_0^{\zeta_g}\sqrt{u_{-,\tau_-}^2 - n_{-,\tau_-}^2}\, d\tau_- = s_0\int\limits_0^{\zeta_g} d\Omega_-, \ (21)$$

$$S_+ = s_0\int\limits_{\zeta_g}^{\infty}\sqrt{u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2}\, d\zeta_+ = s_0\int\limits_{\zeta_g}^{\infty} d\Omega_+. \ (22)$$

Here $d\Omega_\pm^2 = du_\pm^2 - dn_\pm^2 > 0$ are metrics on the minisuperspaces. Using the gauge condition $N = 1$, the constrains can be rewritten in the form

$$u_{-,\tau_-}^2 - n_{-,\tau_-}^2 = 1, \quad u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2 = 1\,. \qquad (23)$$

Therewith the initial actions (9) we reduced to the actions (21) and (22) for the geodesic equations in the minisuperspaces.

2.2. Hamiltonian and general solution of Einstein equations

From the Lagrangian (15) where $N_- = 1$, we obtain the momentums and in T-region

$$P_{u_-} = s_0 u_{-,\tau_-}, \qquad P_{n_-} = -s_0 n_{-,\tau_-}, \qquad (24)$$

$$H_-(\dot{q}, q) = \frac{s_0}{2}\left(u_{-,\tau_-}^2 - n_{-,\tau_-}^2 - 1\right), \qquad (25)$$

$$H_-(p, q) = \frac{1}{2s_0}\left(P_{u-}^2 - P_{n-}^2\right) - \frac{s_0}{2}\,. \qquad (26)$$

In R-region, by analogy, from (17) we obtain

$$P_{u_+} = s_0 u_{+,\zeta_+}, \qquad P_{n_+} = -s_0 n_{+,\zeta_+}, \qquad (27)$$

$$H_+(q', q) = \frac{s_0}{2}\left(u_{+,\zeta_+}^2 - n_{+,\zeta_+}^2 - 1\right), \qquad (28)$$

$$H_+(p, q) = \frac{1}{2s_0}\left(P_{u_+}^2 - P_{n_+}^2\right) - \frac{s_0}{2}\,, \qquad (29)$$

where an evolutional coordinate is the spacelike coordinate ζ_+.

Easily to see that momentums $P_{u_\pm}, P_{n_\pm}$ are saved, and Hamiltonians $H_\pm$ are vanished by virtue of the constrains (23). Hence we obtain the general solutions of geodesic equations on the minisuperspaces in the R- and T-regions

$$u_- = \frac{1}{s_0} P_{u-}\tau_- + C_{u_-},\ n_- = -\frac{1}{s_0} P_{n-}\tau_- + C_{n_-}, \ (30)$$

$$u_+ = \frac{1}{s_0} P_{u_+}\zeta_+ + C_{u_+},\ n_+ = -\frac{1}{s_0} P_{n_+}\zeta_+ + C_{n_+}, \ (31)$$

$$R_+ = \frac{r}{s_0}\left(P_{u_+} - P_{n_+}\right) + \frac{1}{2} R_g\left(C_{u_+} + C_{n_+}\right), \qquad (32)$$

where momentums obey the constrains

$$P_{u-}^2 - P_{n-}^2 = P_{u+}^2 - P_{n+}^2 = s_0^2. \qquad (33)$$

If one substitutes these expression for $u_\pm$ in metrics (14,16), and taking into account constrains (33), one gets the general solutions Einstein equations in the R- and T-regions for the calibrations $N = 1$.

2.3. Matching conditions

Since the surface $R(r, x^0) = R_g$ divides ST on the T- and R-regions with the metrics (14) and (16), then there is a problem finding of the matching conditions. Foremost the first quadratic forms of the sections $\tau = \zeta = const$ must be equal.

$$\left(ds_-^2\right)_{\tau_-=\zeta_g} = \left(ds_+^2\right)_{\zeta_+=\zeta_g} \quad \tau_- = \zeta_+ = \zeta_g, \qquad (34)$$

From here, supposing $\zeta_- = \tau_+$, we obtain

$$u_- = u_+, \qquad n_- = n_+ \text{ at } \tau_- = \zeta_+ = \zeta_g.$$

Now, we consider variations of the action S at $N = 1$:

$$\delta S = \delta S_- + \delta S_+ = \int\limits_0^{\zeta_g} \delta L_- d\tau_- + \int\limits_{\zeta_g}^{\infty} \delta L_+ d\zeta_+ . \qquad (35)$$

From here, at the fixed boundary conditions we obtain the motion equations in the T- and R-regions:

$$u_{-,\tau_-\tau_-} = 0, \quad n_{-,\tau_-\tau_-} = 0\,, \qquad (36)$$

$$n_{+,\zeta_+\zeta_+} = 0, \quad u_{+,\zeta_+\zeta_+} = 0\,. \qquad (37)$$

Let now the fields on the infinity and in the center are fixed, and on the boundary of the T- and R-regions they are free. Then using (35) we find

$$\delta S = s_0 \left[\left(u_{+,\zeta_+} - u_{-,\tau_-}\right)\delta u_- - \right. \qquad (38)$$

$$\left. - \left(n_{-,\tau_-} - n_{+,\zeta_+}\right)\delta n_-\right]_{\tau=\zeta=\zeta_g} . \qquad (39)$$

By virtue of the $\delta S = 0$ and arbitrariness of value $\{\delta u_-, \delta n_-\}$ from here follows the matching conditions of the derivatives. As a result, we obtain

$$u_- = u_+, \quad n_- = n_+, \qquad (40)$$

$$u_{-,\tau_-} = u_{+,\zeta_+}, \quad n_{-,\tau_-} = n_{+,\zeta_+} . \qquad (41)$$

2.4. Mass function in the R- and T-regions

From the definition of the mass function

$$M = \frac{c^2}{2\kappa} R\left(1 + \gamma^{ab} R_{,a} R_{,b}\right) \qquad (42)$$

we obtain the mass functions in the T- and R-regions as the functions of the velocities and coordinates:

$$M_- = \frac{c^2 R_g}{4\kappa}\left(n_- + u_- + (n_- - u_-)\left(n_{-,\tau_-} + u_{-,\tau_-}\right)^2\right), \qquad (43)$$

$$M_+ = \frac{c^2 R_g}{4\kappa}\left(u_+ + n_+ - (u_+ - n_+)\left(u_{+,\zeta_+} + n_{+,\zeta_+}\right)^2\right). \qquad (44)$$

or, as the functions of the momentums and coordinates

$$M_- = \frac{c^2 R_g}{4\kappa}\left(n_- + u_- + \frac{1}{s_0^2}(n_- - u_-)\left(P_{u_-} - P_{n_-}\right)^2\right), \qquad (45)$$

$$M_+ = \frac{c^2 R_g}{4\kappa}\left(u_+ + n_+ - \frac{1}{s_0^2}(u_+ - n_+)\left(P_{u_+} - P_{n_+}\right)^2\right), \qquad (46)$$

Using the motion equations or constrains, it is easy to show that the derivatives of the mass functions with respect to evolutional coordinates τ_-, or ζ_+ in the T- or R-fields, respectively, as well as Poisson brackets between Hamiltonians and mass functions, vanish

$$(M_\pm)_{,\pm} = \{M_\pm, H_\pm\} = \frac{{}^2R_g}{4\kappa s_0^2}\left(P_{u\pm} - P_{n\pm}\right) H_\pm = 0.$$

Here $(M_-)_{,-} = \partial M_-/\partial\tau_-$ and $(M_+)_{,+} = \partial M_+/\partial\zeta_+$. Thus, the dynamical quantities $M_\pm(p, q)$ are the integrals of motion.

2.5. The construction of metrics with the help of constrains and mass functions

Let us assume that $M(q, p) = m$. This implies

$$u_\pm + n_\pm - \frac{1}{s_0^2}\left(u_\pm - n_\pm\right)\left(P_{u_\pm} - P_{n_\pm}\right)^2 = \frac{4\kappa m}{c^2 R_g}.$$

Taking into account the constrains (33) from here we find

$$P_{u_\pm} = \pm\frac{s_0\left(u_\pm - 2\kappa m/c^2 R_g\right)}{\sqrt{\left(u_\pm - 2\kappa m/c^2 R_g\right)^2 - \left(n_\pm - 2\kappa m/c^2 R_g\right)^2}}\,, \qquad (47)$$

$$P_{n_\pm} = \mp\frac{s_0\left(n_\pm - 2\kappa m/c^2 R_g\right)}{\sqrt{\left(u_\pm - 2\kappa m/c^2 R_g\right)^2 - \left(n_\pm - 2\kappa m/c^2 R_g\right)^2}}\,. \qquad (48)$$

Further, from the general solutions (30) and (30), we obtain $u(0) = C_u$, $n(0) = C_n$. Hence

$$P_{u_\pm} = \pm\epsilon_2 \frac{s_0\left(C_{u_\pm} - 2\kappa m/c^2 R_g\right)}{\sqrt{\left(C_{u_\pm} - 2\kappa m/c^2 R_g\right)^2 - \left(C_{n_\pm} - 2\kappa m/c^2 R_g\right)^2}}, \qquad (49)$$

$$P_{n_\pm} = -\epsilon_2 \frac{s_0\left(C_{n_\pm} - 2\kappa m/c^2 R_g\right)}{\sqrt{\left(C_{u_\pm} - 2\kappa m/c^2 R_g\right)^2 - \left(C_{n_\pm} - 2\kappa m/c^2 R_g\right)^2}}. \qquad (50)$$

From the last expression in the (13) it follows that $u_+ = 2R_+/R_g - n_+$. Then from the metric (16), using the (8) and (13), we find that $g_{00} = 1 - n_+ R_g/R_+$. Owing to asymptotic condition $g_{00} = 1 - R_g/R_+$ when $R_+ \to \infty$, where $R_g = 2\kappa m/c^2$, it follows from this $n_+ \to 1$ and $s_0 = \kappa m^2/c$. Then from (31) we have $P_{n_+} = 0$, $C_{n_+} = 1$ and momentum constrain (33) gives $P_{u_+} = s_0$. Here we confined oneself to a positive value. As a result, the solution (31), takes the form $u_+ = \zeta_+ + C_{u_+}, n_+ = 1$. Assumed that that $C_{u_+} = 0$ and substituting the obtained solution in (16), where $N_+ = 1$, we obtain the Schwarzschild solution in a scale-invariant form

$$ds_+^2 = \frac{R_g^2}{4}\left\{\frac{\zeta_+ - 1}{\zeta_+ + 1} d\tau_+^2 - \frac{\zeta_+ + 1}{\zeta_+ - 1} d\zeta_+^2 - (\zeta_+ + 1)^2\, d\sigma^2\right\}, \quad (51)$$

Finally, using (32), we find $R = r + \kappa m/c^2$. Taking into account the relations $u_+ = \zeta_+, n_+ = 1$, (8) and (16) we come to the standard expression for the Schwarzschild metric.

In order to find the metric in the T-region we use the solution (30), the matching conditions (40), and the metric (14). As a result we obtain a trajectory of the system $\{u_- = \tau,\ n_- = 1\}$ in the minisuperspace and the metric of ST in the scale-invariant form

$$ds_-^2 = \frac{R_g^2}{4}\left\{\frac{1 + \tau_-}{1 - \tau_-} d\tau_-^2 - \frac{1 - \tau_-}{1 + \tau_-} d\zeta_-^2 - (1 + \tau_-)^2\, d\sigma^2\right\}. \quad (52)$$

3. Quantum description

3.1. Quantization in the T-region

When choosing a method of quantization by the formula $\hat{H} \sim \Delta + qR$ (choice of the order for operators) we can use the usual covariant quantization, assuming $\hat{H} \sim \Delta$, because minisuperspace is flat. With this in mind, we have transformed the classical action and the metric of minisuperspace to the maximum simple Lorentzian form. Here evolution variable is the dimensionless timelike coordinate τ, the generalized coordinates and velocities are $\{u_-, n_-\}$ and $\{u_{-,\tau_-},\ n_{-,\tau_-}\}$, the momentums are $\{P_{u-},\ P_{n-}\}$. We define the momentum operators by the standard formulae

$$\hat{P}_{n_-} = -i\hbar\frac{\partial}{\partial n_-}, \qquad \hat{P}_{u_-} = -i\hbar\frac{\partial}{\partial u_-}. \quad (53)$$

Then the Hamiltonian has the form

$$\hat{H}_- = -\frac{\hbar^2 c}{2\kappa m^2}\left(\frac{\partial^2}{\partial u_-^2} - \frac{\partial^2}{\partial n_-^2}\right) - \frac{\kappa m^2}{2c}. \quad (54)$$

The Hamilton constrain leads to the DeWitt equation

$$\hat{H}_-\Psi_- = 0 \Rightarrow \left(\frac{\partial^2}{\partial u_-^2} - \frac{\partial^2}{\partial n_-^2}\right)\Psi_- + \mu^4\Psi_- = 0, \quad (55)$$

where $\mu = \sqrt{s_0/\hbar} = m/m_{pl}$, $m_{pl} = \sqrt{c\hbar/\kappa}$. The mass function in the T-region corresponds to the mass operator

$$\hat{M}_- = \frac{m}{2}\left(u_- + n_- + \frac{u_- - n_-}{\mu^4}\left(\frac{\partial}{\partial u_-} - \frac{\partial}{\partial n_-}\right)^2\right). \quad (56)$$

Further, we introduce the light coordinates in the minisuperspace of the T-region

$$\xi_- = u_- - n_-, \qquad \eta_- = u_- + n_-. \quad (57)$$

Then the Hamiltonian, DeWitt equation and mass operator take the forms

$$\hat{H}_- = -\frac{2c\hbar^2}{\kappa m^2}\frac{\partial^2}{\partial\xi_-\partial\eta_-} - \frac{\kappa m^2}{2c}, \quad (58)$$

$$\frac{\partial^2\Psi_-}{\partial\xi_-\partial\eta_-} + \frac{\mu^4}{4}\Psi_- = 0, \quad (59)$$

$$\hat{M}_- = \frac{m}{2}\left(\eta_- + \frac{4}{\mu^4}\xi_-\frac{\partial^2}{\partial\xi_-^2}\right). \quad (60)$$

We consider the operator commutator

$$\left[\hat{H}_-\hat{M}\right]\Psi_- = \frac{4}{\mu^4}\hat{H}_-\frac{\partial\Psi_-}{\partial\xi_-} = \frac{4}{\mu^4}\frac{\partial}{\partial\xi_-}\left(\hat{H}_-\Psi_-\right), \quad (61)$$

We see that commutator vanishes on constrain $\hat{H}_-\Psi_- = 0$. Therefore, the eigenvalue problem of the mass operator: $\hat{M}_-\Psi_- = m\Psi_-$ is necessary to solve together with the finding of the wave functions, satisfying DeWitt equation. Thus the need to solve together a system of equations

$$\frac{\partial^2\Psi_-}{\partial\xi_-\partial\eta_-} = -\frac{\mu^4}{4}\Psi_-, \quad \mu^4 = \frac{m^4}{m_{pl}^4}, \quad (62)$$

$$\frac{\partial^2\Psi_-}{\partial\xi_-^2} = -\frac{\mu^4}{4}\left(\frac{\eta_- - 2}{\xi_-}\right)\Psi_-. \quad (63)$$

The compatibility condition leads to the equation

$$\xi_-\frac{\partial\Psi_-}{\partial\xi_-} + (2 - \eta_-)\frac{\partial\Psi_-}{\partial\eta_-} = \Psi_-. \quad (64)$$

Hence we find the $\Psi_- = \xi_- G(Z)$, where $G(Z)$ is the arbitrary function of argument $Z = \xi_-(2 - \eta_-)$. Substituting the Ψ_- into DeWitt equation (62), we get

$$Z\frac{d^2G}{dZ^2} + 2\frac{dG}{dZ} - \frac{\mu^4}{4}G = 0. \quad (65)$$

Its general solution

$$G = \frac{1}{\sqrt{Z}}\left\{C_1 J_1\left(\sqrt{-\mu^4 Z}\right) + C_2 Y_1\left(\sqrt{-\mu^4 Z}\right)\right\}. \quad (66)$$

where $J_1(x)$ and $Y_1(x)$ are the Bessel functions of the first and second kind. Thus, the common wave function of the Hamiltonian and the mass operator takes the form

$$\Psi_- = \frac{\xi_-}{\sqrt{Z_-}}\left\{C_1 J_1\left(\mu^2\sqrt{-Z_-}\right) + C_2 Y_1\left(\mu^2\sqrt{-Z_-}\right)\right\}. \quad (67)$$

Returning to the variables $\{u_-,\ n_-\}$, we find

$$\Psi_- = \frac{u_- - n_-}{\sqrt{(n_- - 1)^2 - (u_- - 1)^2}} \cdot$$

$$\cdot\left\{C_1 J_1\left(\mu^2\sqrt{(u_- - 1)^2 - (n_- - 1)^2}\right) + \right.$$

$$\left. + C_2 Y_1\left(\mu^2\sqrt{(u_- - 1)^2 - (n_- - 1)^2}\right)\right\}.$$

The minisuperspace metric $d\Omega_-^2 = du_-^2 - dn_-^2 > 0$ in the T-region gives the physically admissible timelike directions $(u_- - 1)^2 - (n_- - 1)^2 > 0$ of the vector $\xi_-^a = u_- - 1,\ n_- - 1$. For the regularity of the wave function, on the light-cone $(u_- - 1)^2 - (n_- - 1)^2 = 0$ in the minisuperspace, we suppose that $C_2 = 0$. As a result, the wave function of the black hole with the mass m in T-region takes the form:

$$\Psi_- = \frac{C_-(u_- - n_-)J_1\left(\mu^2\sqrt{(u_- - 1)^2 - (n_- - 1)^2}\right)}{\sqrt{(n_- - 1)^2 - (u_- - 1)^2}}. \quad (68)$$

The found wave function depends on the square two-dimensional vector $\xi_-^a = \{u_- - 1, n_- - 1\}$ with initial point $\{1,1\}$. In the physical region the wave function oscillates and decreases. Outside the light cone, it decreases monotonically

3.2. Quantization in the R-region

Here the formal evolution variable is the dimensionless spacelike coordinate ζ, the generalized coordinates and velocities are $\{n_+, u_+\}$ $\{u_{+,\zeta_+},\ n_{+,\zeta_+}\}$, the momentums are $\{P_{u_+}, P_{n_+}\}$. We define formally momentum operators by the formulae

$$\hat{P}_{u_+} = -i\hbar\frac{\partial}{\partial u_+},\quad \hat{P}_{n_+} = -i\hbar\frac{\partial}{\partial n_+}. \quad (69)$$

The Hamiltonian has the form

$$\hat{H}_+ = -\frac{c\hbar^2}{2\kappa m^2}\left(\frac{\partial^2}{\partial u_+^2} - \frac{\partial^2}{\partial n_+^2}\right) - \frac{\kappa m^2}{2c}, \quad (70)$$

The Hamilton constrain $H_+ = 0$ leads to the DeWitt equation:

$$\hat{H}_+\Psi_+ = 0 \Rightarrow \left(\frac{\partial^2}{\partial u_+^2} - \frac{\partial^2}{\partial n_+^2}\right)\Psi_+ + \mu^4\Psi_+ = 0. \quad (71)$$

The mass function corresponds to the mass operator

$$\hat{M}_+ = \frac{m}{2}\left(u_+ + n_+ + \frac{u_+ - n_+}{\mu^4}\left(\frac{\partial}{\partial u_+} - \frac{\partial}{\partial n_+}\right)^2\right) \quad (72)$$

We introduce the light coordinates in the minisuperspace of the R-region

$$\xi_+ \ = \ n_+ - u_+, \qquad \eta_+ = n_+ + u_+\,. \quad (73)$$

Then the Hamiltonian, DeWitt equation and mass operator take the forms

$$\hat{H}_+\Psi_+ = \frac{2c\hbar^2}{\kappa m^2}\frac{\partial^2\Psi_+}{\partial\eta_+\partial\xi_+} - \frac{\kappa m^2}{2c}\Psi_+\,, \quad (74)$$

$$\frac{\partial^2\Psi_+}{\partial\eta_+\partial\xi_+} - \frac{\mu^4}{4}\Psi_+ = 0\,. \quad (75)$$

$$\hat{M}_+ = \frac{m}{2}\left(\eta_+ - \frac{4}{\mu^4}\xi_+\frac{\partial^2}{\partial\xi_+^2}\right). \quad (76)$$

The commutator of the Hamilton and the mass operator

$$\left[\hat{H}_+\hat{M}\right]\Psi_+ = \frac{4}{\mu^4}\hat{H}_+\frac{\partial\Psi_+}{\partial\xi_+} = \frac{4}{\mu^4}\frac{\partial}{\partial\xi_+}\left(\hat{H}_+\Psi_+\right). \quad (77)$$

vanishes on the constrain $\hat{H}_+\Psi_+ = 0$. Therefore, the eigenvalue problem of the mass operator: $\hat{M}_+\Psi_+ = m\Psi_+$ should be solved together with the finding of the wave functions, satisfying DeWitt equation. So necessary to solve together a system of equations

$$\frac{\partial^2\Psi_+}{\partial\eta_+\partial\xi_+} = \frac{\mu^4}{4}\Psi_+, \quad (78)$$

$$\frac{\partial^2\Psi_+}{\partial\xi_+^2} = \frac{\mu^4}{4}\left(\frac{\eta_+ - 2}{\xi_+}\right)\Psi_+. \quad (79)$$

The compatibility condition leads to the equation

$$\xi_+\frac{\partial\Psi_+}{\partial\xi_+} + (2 - \eta_+)\frac{\partial\Psi_+}{\partial\eta_+} = \Psi_+, \quad (80)$$

From here we find that $\Psi_+ = \xi_+ G_+(\tilde{Z})$, where $\tilde{Z} = \xi_+(2 - \eta_+)$. Substituting this relation into DeWitt equation we derive the equation for the function $G_+(\tilde{Z})$:

$$\tilde{Z}\frac{d^2G_+}{d\tilde{Z}^2} + 2\frac{dG_+}{d\tilde{Z}} + \frac{\mu^4}{4}G_+ = 0. \quad (81)$$

Its general solution taking into account the formula $\Psi_+ = \xi_+ G_+(\tilde{Z})$ leads to the wave function of the state with a certain mass

$$\Psi_+ = \frac{\xi_+}{\sqrt{\tilde{Z}}}\left\{C_1 J_1\left(\mu^2\sqrt{\tilde{Z}}\right) + C_2 Y_1\left(\mu^2\sqrt{\tilde{Z}}\right)\right\}. \quad (82)$$

Take into consideration that $\tilde{Z} = (u_+ - 1)^2 - (n_+ - 1)^2$, from the regularity condition of the wave function on the light cone, we get $C_2 = 0$. As a result, wave function of BH with the mass m in the R-region takes the form:

$$\Psi_+ = \frac{C_+(u_+ - n_+)J_1(\mu^2\sqrt{(u_+ - 1)^2 - (n_+ - 1)^2})}{\sqrt{(u_+ - 1)^2 - (n_+ - 1)^2}}. \quad (83)$$

The found wave function describes the standing decreasing spherical wave in the minisuperspace of the R-region.

4. Conclusion

The resulting wave function (68), (83) satisfy the matching conditions (40). Furthermore in full minisuperspace R- and T- regions, we can enter the general smooth coordinates by the formulae

$$u = \begin{cases} u_- = \eta_-, & -1 < \eta_- < 1 \\ u_+ = \zeta_+, & 1 < \zeta_+ < \infty \end{cases},$$

$$n = \left\{ \begin{array}{ll} n_- = 1, & -1 < \eta_- < 1 \\ n_+ = 1, & 1 < \zeta_+ < \infty \end{array} \right\} = 1.$$

Herewith the common wave function of the BH for the whole space $V^4 = T \cup R$ has the form

$$\Psi = \frac{A(u-n)}{\sqrt{(n-1)^2 - (u-1)^2}} J_1(\mu^2 \sqrt{(u-1)^2 - (n-1)^2}) \,. \quad (84)$$

It corresponds to the state with a certain mass $m = \mu m_{pl}$. The mass spectrum obtained continuously. This is in agreement c other works (See e.g., Cavaglia‘ at all, 1995), where other methods are used. Interesting to note that formal quantization in R-region with spacelike evolution coordinate gives the same wave function that and in the T-region with the timelike evolution coordinate.

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