SCALAR COSMOLOGICAL PERTURBATIONS OF PRESSURELESS MATTER IN THE BRANEWORLD MODEL

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ABSTRACT. We present a complete set of differential equations describing the evolution of scalar cosmological perturbations of pressureless matter on the brane in the general case where the action of the model contains the induced-curvature term as well as the cosmological constants in the bulk and on the brane.

Key words: Extra dimensions; braneworld model; cosmological perturbations.

1. Introduction

The idea that our observable universe can be a four-dimensional manifold (the "brane") embedded in higher dimensional spacetime (the "bulk") with Standard Model particles and fields trapped on the brane was thoroughly investigated during the last two decades. The activity in the field was triggered especially by the Randall–Sundrum (RS) braneworld model (Randall & Sundrum, 1999), in which Einstein's theory of general relativity is modified due to extra dimensional effects at relatively high energies. Apart from interesting cosmological applications, it was shown that a modified theory of gravity based on the RS braneworld model potentially can explain the observations of the galactic rotation curves and X-ray profiles of galactic clusters without invoking the notion of dark matter (see, for example, Mak & Harko, 2004). This theory, however, was unable to address the cosmological implications of dark matter. On the other hand, an alternative braneworld model of Dvali, Gabadadze and Porrati (DGP, Dvali et al., 2000), in which gravity is modified at low energies, gave rise to a cosmology with late-time acceleration without cosmological constants on the brane or in the bulk.

The main feature of the DGP braneworld model is the induced-gravity term in the action for the brane. But cosmological constants are absent in this theory, in contrast to the RS braneworld model. A more general braneworld model contains the induced-gravity term as well as cosmological constants in the bulk and on the brane (Shtanov, 2000). Models of such generic form can describe late-time cosmological acceleration and, in doing so, they exhibit some interesting specific features. At the same time, this kind of braneworld model can be used to address astrophysical observations of dark matter in galaxies (Viznyuk & Shtanov, 2007).

Developing the theory of cosmological perturbations is a long-standing problem of the braneworld model. Regardless the computational complexity of this problem, a considerable progress has been made during the last years. A complete system of cosmological equations allowing for numerical computation was obtained in the framework of the RS (Cardoso et al., 2007) and DGP (Sawicki et al., 2007) braneworld models. Important analytical results are presented in Koyama & Maartens, 2006. However, the problem of cosmological perturbations in the braneworld model still remains to be solved in full generality. For a modern review of this problem, see Maartens & Koyama, 2010.

The existence of extra dimension requires specification of the boundary conditions in the bulk space. In the usual case of a spatially flat brane, the extra dimension is noncompact, and one has to deal with the spatial infinity of the extra dimension. This is a difficult situation with no obvious and unique choice for the boundary conditions. In the present work, we consider the case of a spatially closed brane, an expanding three-sphere, which is bounding a four-ball in the bulk space. The boundary condition for such configuration can be specified uniquely just as a regularity condition of the metric inside the ball.

2. The theory

The braneworld action, to lowest order in the bulk and brane curvature, can be written in the form:

$$S = M^3 \left[\int_{\rm bulk} (\mathcal{R} - 2\Lambda) - 2 \int_{\rm brane} K \right] + \int_{\rm brane} \left(m^2 R - 2\lambda \right) + \int_{\rm brane} L \left(g_{\mu\nu}, \phi \right) , \qquad (1)$$

where $\mathcal{R}$ is the scalar curvature of the five-dimensional bulk metric g_{AB}, and R is the scalar curvature of the induced metric $g_{\mu\nu}$ on the brane. The quantity K denotes the trace of the symmetric tensor of extrinsic curvature of the brane, and the symbol $L(g_{\mu\nu},\phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric $g_{\mu\nu}$. The symbols M and m denote the five-dimensional and four-dimensional Planck masses, respectively, Λ is the bulk cosmological constant, and λ is referred to as brane tension.

Action (1) leads to the Einstein equation with cosmological constant in the bulk,

$$\mathcal{G}_{AB} + \Lambda g_{AB} = 0\,, \tag{2}$$

and the following equation on the brane:

$$G_{\mu\nu} + \frac{\Lambda_{\rm RS}}{b+1} g_{\mu\nu} = \left(\frac{b}{b+1}\right)\frac{1}{m^2}T_{\mu\nu} + \frac{1}{b+1}\left(\frac{1}{M^6}Q_{\mu\nu} - C_{\mu\nu}\right)\,, \tag{3}$$

where $b = k\ell\,,\ k = \lambda/3M^3\,,\ \ell = 2m^2/M^3$ are convenient parameters of the braneworld model, $\Lambda_{\rm RS} = \Lambda/2+3k^2$ is the value of the effective cosmological constant in the Randall–Sundrum model,

$$Q_{\mu\nu} = \frac{1}{3}EE_{\mu\nu} - E_{\mu\lambda}E^{\lambda}{}_{\nu} + \frac{1}{2}\left(E_{\rho\lambda}E^{\rho\lambda} - \frac{1}{3}E^2\right)g_{\mu\nu} \tag{4}$$

is a quadratic expression with respect to the 'bare' Einstein equation $E_{\mu\nu} \equiv m^2G_{\mu\nu} - T_{\mu\nu}$ on the brane, and $E = g^{\rho\lambda}E_{\rho\lambda}$. The symmetric traceless tensor $C_{\mu\nu}$ is the projection of the bulk Weyl tensor $\mathcal{C}_{ABCD}$ which carries information about the gravitational field outside the brane.

The background cosmological evolution on the brane can be presented in the following form (Sahni & Shtanov, 2003):

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho+\lambda}{3m^2} + \frac{2}{\ell^2}\left[1 \pm \sqrt{1+\ell^2\left(\frac{\rho+\lambda}{3m^2} - \frac{\Lambda}{6}\right)}\right]. \tag{5}$$

Here, $\rho = \rho(t)$ is the matter energy density on the brane. The Hubble parameter $H \equiv \dot a/a$ describes the evolution of the Friedmann–Robertson–Walker (FRW) metric $ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^idx^j$, $\kappa = 0, \pm1$ is a spatial curvature of a purely spatial metric γ_{ij}. Here and hereafter, we neglect the influence of symmetryc tensor $C_{\mu\nu}$ for the background geometry.

3. Scalar cosmological perturbations on the brane

Scalar cosmological perturbations of the induced metric on the brane are most conveniently described by the relativistic potentials Φ and Ψ in the so-called longitudinal gauge. The perturbed metric in the conformal coordinates reads

$$ds^2 = a^2\left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\gamma_{ij}dx^idx^j\right]\,. \tag{6}$$

We introduce the components of the linearly perturbed stress–energy tensor of matter in these coordinates:

$$\delta T^{\mu}{}_{\nu} = \begin{pmatrix} -\delta\rho\,, & -\dfrac{\nabla_i v}{a} \\ \dfrac{\nabla^i v}{a}\,, & \delta p\,\delta^i{}_j + \dfrac{\zeta^i{}_j}{a^2} \end{pmatrix}\,, \tag{7}$$

where $\delta\rho$, δp, v, and $\zeta_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)\zeta$ are scalar perturbations. Similarly, we introduce the scalar perturbations $\delta\rho_{\mathcal{C}}$, $v_{\mathcal{C}}$, and $\delta\pi_{\mathcal{C}}$ of the tensor $C_{\mu\nu}$:

$$m^2\delta C^{\mu}{}_{\nu} = \begin{pmatrix} -\delta\rho_{\mathcal{C}}\,, & -\dfrac{\nabla_i v_{\mathcal{C}}}{a} \\ \dfrac{\nabla^i v_{\mathcal{C}}}{a}\,, & \dfrac{\delta\rho_{\mathcal{C}}}{3}\delta^i{}_j + \dfrac{\delta\pi^i{}_j}{a^2} \end{pmatrix}\,, \tag{8}$$

where $\delta\pi_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)\delta\pi_{\mathcal{C}}$.

We call v and $v_{\mathcal{C}}$ the momentum potentials for matter and dark radiation, respectively, $\delta\rho$ and $\delta\rho_{\mathcal{C}}$ are their energy density perturbations, and ζ and $\delta\pi_{\mathcal{C}}$ are the scalar potentials for their anisotropic stresses.

Using this notation and equation (3), one can derive a complete system for perturbations on the brane, which, however, is not closed. To illustrate this problem, we present the result for *pressureless*[1] matter ($p=0$, $\zeta=0$):

$$\ddot\Delta + 2H\dot\Delta = \left(1+\frac{6\gamma}{\beta}\right)\frac{\rho\Delta}{2m^2} + (1+3\gamma)\frac{\delta\rho_{\mathcal{C}}}{m^2\beta}\,, \tag{9}$$

$$\delta\dot\rho_{\mathcal{C}} + 4H\delta\rho_{\mathcal{C}} = \frac{1}{a^2}\nabla^2 v_{\mathcal{C}}\,, \tag{10}$$

$$\dot v_{\mathcal{C}} + 4Hv_{\mathcal{C}} = \gamma\Delta_{\mathcal{C}} + \left(\gamma - \frac{1}{3}\right)\rho\Delta + \frac{4}{3(1+3\gamma)a^2}\left(\nabla^2 + 3\kappa\right)\delta\pi_{\mathcal{C}}\,, \tag{11}$$

where $\Delta \equiv (\delta\rho + 3Hv)/\rho$ is a conventional dimensionless variable describing the matter perturbations, and, similarly, $\Delta_{\mathcal{C}} \equiv \delta\rho_{\mathcal{C}} + 3Hv_{\mathcal{C}}$. The time-dependent dimensionless functions β and γ are given by

$$\beta \equiv \pm 2\ell\sqrt{\left(\frac{\rho+\lambda}{3m^2} + \frac{1}{\ell^2} - \frac{\Lambda}{6}\right)}\,, \tag{12}$$

$$\gamma \equiv \frac{1}{3}\left[1 - \frac{\dfrac{\rho}{2m^2}}{\left(\dfrac{\rho+\lambda}{3m^2} + \dfrac{1}{\ell^2} - \dfrac{\Lambda}{6}\right)}\right]\,. \tag{13}$$

The system of equations (9)–(11) is not closed because it does not contain an evolution equation for the anisotropic stress $\delta\pi_{\mathcal{C}}$. The evolution of the Weyl tensor should be derived from the perturbed

[1] The case of more general matter is considered in Viznyuk & Shtanov, 2012.

bulk equation (2) after setting some natural boundary conditions in the bulk (Mukohyama, 2001).

4. Scalar perturbation of the bulk metric

To find the perturbed Weyl tensor, we consider a spatially closed braneworld model ($\kappa = 1$) which bounds the interior of flat ($\Lambda = 0$) bulk space with the spatial topology of a ball and demand that the bulk metric be regular in the brane interior. In the natural static coordinates, the background bulk metric can be written in this case in the form

$$ds^2_{\rm bulk} = -d\tau^2 + dr^2 + r^2\gamma_{ij}dx^i dx^j\,, \qquad (14)$$

where γ_{ij} is the metric of a 3–dimensional maximally symmetric space with coordinates x^i. In these coordinates, the FRW brane moves radially along the trajectory $r = a(\tau)$, and the relevant part of the bulk is given by $r \le a(\tau)$.

The components of the perturbed bulk Weyl tensor $\delta\mathcal{C}_{ABCD}$ and it's projection to the brane $\delta C_{\mu\nu}$ can be expressed in terms of the Mukohyama master variable in the following way:

$$\frac{\delta\rho_{\mathcal{C}}}{m^2} = -\frac{n(n+2)(n^2+2n-3)}{3a^5}\,\Omega_{\rm b}\,, \qquad (15)$$

$$\frac{v_{\mathcal{C}}}{m^2} = \frac{(n^2+2n-3)}{3a^3}\Big[aH\left(\partial_r\Omega\right)_{\rm b} - H\Omega_{\rm b} + $$
$$+\sqrt{1+a^2H^2}\left(\partial_\tau\Omega\right)_{\rm b}\Big]\,, \qquad (16)$$

$$\frac{\delta\pi_{\mathcal{C}}}{m^2} = -\frac{\left(1+2a^2H^2\right)}{2a}\left(\partial_\tau^2\Omega\right)_{\rm b} -$$
$$-H\sqrt{1+a^2H^2}\left(\partial^2_{\tau r}\Omega\right)_{\rm b} - \frac{\left(1+3a^2H^2\right)}{2a^2}\left(\partial_r\Omega\right)_{\rm b} -$$
$$-\frac{(n^2+2n-3)\left(1+3a^2H^2\right)}{6a^3}\,\Omega_{\rm b}\,, \qquad (17)$$

where, $a = a(t)$ is a scale factor of the background Friedmann–Robertson–Walker metric on the brane, $H = \dot{a}/a$ is the Hubble parameter on the brane, and the function $\tau = \tau(t)$ is defined by the differential equation $d\tau/dt = \sqrt{1+a^2H^2}$. The subscript $\{\}_{\rm b}$ means that the value of the corresponding quantity is taken at the brane. For example, $\Omega_{\rm b}(t) \equiv \Omega[\tau(t), a(t)]$. Integers n numerates the discrete Laplacian eigenvalues on the three-sphere: $k_n^2 = n(n+2)$, $n = 0, 1, 2\ldots$

Mukohyama master variable $\Omega(\tau, r)$ in our case can be determined as a solution of equation (Mukohyama, 2000):

$$-\partial_\tau^2\Omega + \partial_r^2\Omega - \frac{3}{r}\partial_r\Omega - \frac{(n^2+2n-3)}{r^2}\Omega = 0\,, \qquad (18)$$

which can be solved in terms of special functions once the regulatory conditions in the bulk are imposed.

5. Concluding remarks

The problem of finding closed system of equations, describing the evolution of cosmological perturbations in the general braneworld model with induced curvature and cosmological constants, appears to be solved. Basic equation in the bulk (18) complemented with the boundary conditions on the brane (9–11), together with (15–17), represent the desired result for pressureless matter.

Having this result in mind, one can act in two possible ways: to develop numerical method for computation similar to those employed in the analysis of perturbations in the RS (Cardoso et al., 2007) and DGP (Sawicki et al., 2007) braneworld models, or to develop analytic approximate methods similar to the quasi-static approximation (Koyama & Maartens, 2006). Both are the subject of further investigation.

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References

Cardoso A., Hiramatsu T., Koyama K., Seahra S.S.: 2007, *J. Cosmol. Astropart. Phys.*, **07**, 008.
Dvali G., Gabadadze G., Porrati M.: 2000, *Phys. Lett. B*, **485**, 208.
Koyama K., Maartens R.: 2006, *J. Cosmol. Astropart. Phys.*, **01**, 016.
Maartens R., Koyama K.: 2010, *Living Rev. Rel.*, **13**, 5.
Mak M. K., Harko T.: 2004, *Phys. Rev. D*, **70**, 024010.
Mukohyama S.: 2000, *Phys. Rev. D*, **62**, 084015.
Mukohyama S.: 2001, *Phys. Rev. D*, **64**, 064006.
Randall L., Sundrum R.: 1999, *Phys. Rev. Lett.*, **83**, 4690.
Sahni V,. Shtanov Yu.: 2003, *J. Cosmol. Astropart. Phys.*, **11**, 014.
Sawicki I., Song Y. S., Hu W.: 2007, *Phys. Rev. D*, **75**, 064002.
Shtanov Yu.: 2000, *arXiv*: hep-th/0005193.
Viznyuk A., Shtanov Yu.: 2007, *Phys. Rev. D*, **76**, 064009.
Viznyuk A., Shtanov Yu.: 2012, *Ukr. Phys. J.* (in press).