DETERMINATION OF THE POSITION AND VELOCITY OF THE NATURAL SATELLITE'S SHADOW ON THE ILLUMINATED PART OF THE SPHERICAL PLANET'S VISIBLE DISK

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ABSTRACT. This study introduces a simple analytical method for determination of the rectangular coordinates of the natural satellite's (or the moon's) shadow cast on the illuminated part of the spherical planet's visible disk and projected onto the sky plane, as well as the rates of change of those coordinates when the shadow is moving. The analytical method's formulae, derived in two approximations of the Earth's and the Sun's positions relative to the planet's rotation axis, provide an accurate solution for each approximation at any phase angle values that enables to apply them to simulate processes associated with the satellite's shadow passing across the illuminated part of the planet's visible disk.

Key words: natural satellite, moon, shadow.

1. Introduction

The simulation of the planet satellites' motion provides for the observations of their exact positions. When a natural satellite is orbiting the planet, a geocentric observer can see the satellite's position in orbit at a given instant projected onto the geocentric sky plane relative to the planet's visible disk limb, which is the planets' projection onto the same plane. The geocentric sky plane is perpendicular to the line connecting the centres of the planet and the Earth.

Illumination of the planet and its satellite by the Sun specifies the line that connects the centres of the planet and the Sun. The planet and its natural satellite cast their shadows in the same direction. The satellite's position relative to the planet's shadow and the satellite's shadow position with respect to the planet's disk are determined for a heliocentric observer who can see the satellite's position in orbit at a given instant projected onto the heliocentric sky plane with relation to the planet's terminator, which is the planet's projection onto the same plane. The heliocentric plane of the sky is perpendicular to the line connecting the centres of the planet and the Sun.

When the satellite passes near those two lines (i.e. the planet-Earth and the planet-Sun lines) while orbiting the planet, the satellite-planet phenomena can be observed from the Earth's centre. At that an occultation of the satellite by the planet's disk is observed close to superior geocentric conjunctions, and the satellite's transit in front of the planet's disk can be seen close to inferior geocentric conjunctions. The satellite's eclipse is observed close to superior heliocentric conjunctions, and the satellite's shadow passing across the planet's disk is seen close to inferior heliocentric conjunctions.

The prediction of phenomena in the satellite systems of different planets is based on the determination of the relative positions of the planet's and each of its satellites' projections onto either of the sky planes (the geocentric or heliocentric ones) (Emel'yanov, 1996; Bureau Des Longitudes, 2001). The phenomena conditions are determined independently for each of those sky planes.

The satellite's shadow passing across the planet's disk when the satellite and its shadow can be observed simultaneously is of special interest. In that case, the determination of both the instant of the satellite's shadow contact with the illuminated part of the planet's disk and the shadow's rectangular coordinates in the heliocentric sky plane is carried out similarly to the method described in (Mikhalchuk, 2007). The natural satellite's shadow projection can be seen by a terrestrial observer in the geocentric plane of the sky; however, the determination of the satellite's shadow rectangular coordinates relative to the planet's visible disk in that plane gets complicated due to the satellite's shadow being projected onto the illuminated part of the spherical planet's surface rather than on the heliocentric sky plane (as it is the case when the contact conditions are estimated). Moreover, due to the influence of the phase and spherical surface curvature, the motion of the satellite's shadow across the illuminated part of the planet's visible disk differs from that of the satellite's projection onto the heliocentric sky plane.

The target problem can be solved with the numerical method that implies various coordinate conversions; but such a method is very cumbersome and does not give formulae for immediate determination of the satellite's shadow coordinates and their rates of change.

The indicated features of the natural satellite's shadow position and motion across the illuminated part of the planet's visible disk necessitate the development of an analytical method that provides some formulae to immediately transform the satellite's position projected onto the heliocentric sky plane to its shadow's projection onto the geocentric sky plane. The formulae that describe the satellite's shadow position and motion across the planet's visible disk can be applied to easily and accurately approximate not only the shadow's position and velocity, but also its visible size and shape (Mikhalchuk, 2004), which are quite important when conducting physical, astrometric and photometric observations.

The purpose of this study is to determine the position of the spherical planet satellite's shadow in the geocentric sky plane, as well as its rate of change, using the described analytical method.

2. The determination of the satellite's shadow position on the planet's surface

The formulated problem is solved using the orthographic approximation, i.e. the orthographic limb is the boundary of the planet's visible disk, and the orthographic terminator is the boundary of the illuminated part of the planet's visible disk.

The natural satellite's shadow position on the illuminated part of the planet's visible disk depends on the lighting conditions, i.e. the planet's phase. If the phase angle Φ is nonzero, the position of the satellite's shadow projected onto the geocentric sky plane relative to the planet's visible disk centre does not coincide with the satellite's position projected either onto the same plane or on the heliocentric sky plane.

The planet's orientation in space is specified by the direction of its rotation axis. The apparent position of the planet's rotation axis relative to a terrestrial observer can be defined by two angles, namely the planetocentric declination of the Earth $D_\oplus$ and the angle P of the axis position in the geocentric celestial sphere. The planet's visible disk illuminance relative to a terrestrial observer is defined by the phase angle Φ and the angle Q of the minimum illuminance point position. Those angles are related to the difference $\Delta A = A_\oplus - A_\odot$ of the planetocentric right ascensions of the Earth and the Sun, which can be derived by the following formula adopted from (Mikhalchuk, 2007a):

$$\tan \Delta A = \frac{\sin \Phi \sin(P - Q)}{\cos \Phi \cos D_\oplus + \sin \Phi \sin D_\oplus \cos(P - Q)}. \quad (1)$$

The formula that connects angles Φ, P, Q and $D_\oplus$ (the planet's physical ephemerides) with the planetocentric declination of the Sun $D_\odot$ is given in (Mikhalchuk, 2007a).

To determine the satellite's shadow position observed on the illuminated part of the planet's visible disk, we introduce a rectangular coordinate system (x, y, z), that is oriented relative to the orthographic limb plane (the geocentric plane of the sky). The origin of this coordinate system is in the centre of the planet's geometric disk, and the X and Y axes lie in the sky plane. The Y-axis is oriented along the planet's rotation axis projected onto the sky plane towards its North Pole; the X-axis is oriented eastward; and the Z-axis is oriented along the line of sight outward from the Earth. Let us take the planet's equatorial radius a as a unit for measuring distance. The satellite's apparent position in the geocentric sky plane at a given instant is specified by the rectangular coordinates x and y, and the apparent position of its shadow is specified by the rectangular coordinates x_s and y_s. The velocity V of the planet's satellite motion in the geocentric sky plane close to the satellite's inferior geocentric conjunction has projections onto the X and Y axes as such $\dot{x}$ and $\dot{y}$, respectively; and the projected velocity V_s of its shadow's motion is specified by $\dot{x}_s$ and $\dot{y}_s$. The rectangular coordinates x and y, as well as their rates of change $\dot{x}$ and $\dot{y}$, can be defined by formulae given in (Mikhalchuk, 2007).

Let us introduce another rectangular coordinate system (x', y', z') that is oriented relative to the orthographic terminator plane (the heliocentric plane of the sky). The origin of this coordinate system is in the centre of the planet's geometric disk that can be observed from the Sun, and the X' and Y' axes lie in the orthographic terminator plane. The Y'-axis is oriented along the planet's rotation axis projected onto that plane towards the North Pole, the X'-axis is oriented eastward; and the Z'-axis is oriented along the line of sight outward from the Sun. The satellite's rectangular coordinates x' and y' at a given instant specify the satellite's shadow position in the heliocentric plane of the sky, and the position of its shadow on the planet's surface relative to the same plane is determined by the rectangular coordinates x'_s, y'_s and z'_s. The projections of the satellite and its shadow onto the heliocentric plane of the sky coincides, therefore $x'_s = x'$ and $y'_s = y'$. The velocity V' of the planet's satellite motion in the heliocentric sky plane close to the satellite's inferior heliocentric conjunction has projections onto the X' and Y' axes as such $\dot{x}'$ and $\dot{y}'$, respectively. The method for computing the rectangular coordinates x' and y', as well as their rates of change $\dot{x}'$ and $\dot{y}'$, is described in the study (Mikhalchuk, 2007).

If the planet's satellite is not a close one, i.e. its radius vector r fulfills condition $r \gg a$, its motion projected onto the geocentric and heliocentric sky planes close to its geocentric and heliocentric conjunctions can be deemed uniform, meaning that velocities $\dot{x}$ and $\dot{y}$, as well as $\dot{x}'$ and $\dot{y}'$ are constant.

Hence, based on the known rectangular coordinates x' and y' of the planet's satellite position in the heliocentric sky plane close to its inferior heliocentric conjunction, as well as on their rates of change $\dot{x}'$ and $\dot{y}'$, it is necessary to find the satellite's shadow rectangular coordinates x_s and y_s, as well as their rates of change $\dot{x}_s$ and $\dot{y}_s$.

The methods for accurate solution of the formulated problem, which are initially intended for the spherical planets, can be also applied for its approximate solution for ellipsoid planets. It can be shown that the planet's flattening can be neglected for low ratios $\frac{|y'|}{a}$ and $\frac{|\dot{y}'|}{V'}$ meaning that the planet is spherical.

As in the general case ${x'_s}^2 + {y'_s}^2 + {z'_s}^2 = {x'}^2 + {y'}^2 + {z'_s}^2 = a^2$, an auxiliary dimensionless coordinate should be introduced to solve the target problem:

$$\zeta' = -\frac{z'_s}{a} = \sqrt{1 - \frac{x'^2 + y'^2}{a^2}}, \tag{2}$$

this coordinate should always be positive on the planet's side illuminated by the Sun. The auxiliary coordinate $\zeta' = 0$ on the terminator, and $\zeta' = 1$ at the subsolar point. Having this coordinate differentiated with respect to time, its rate of change can be found:

$$\dot{\zeta}' = -\frac{\dot{z}'_s}{a} = -\frac{x'\dot{x}' + y'\dot{y}'}{a^2\zeta'}. \tag{3}$$

The natural satellite's shadow position relative to the planet's visible disk and its velocity can be defined by transforming the satellite's position projected onto the heliocentric sky plane to its shadow's projection onto the geocentric sky plane.

3. The numerical method for determination of the satellite's shadow position and velocity on the illuminated part of the planet's visible disk

The numerical method enables to find the satellite's shadow rectangular coordinates x_s and y_s in the geocentric sky plane, as well as their rates of change $\dot{x}_s$ and $\dot{y}_s$ by the initial rectangular coordinates of the satellite x' and y' and their rates of change $\dot{x}'$ and $\dot{y}'$ in the heliocentric sky plane. This method can also be implemented with the known formulae for determination of the satellite's shadow position on the illuminated side of the spherical planet (Mikhailov, 1954).

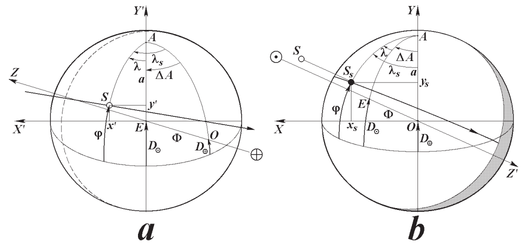

Figure 1: The appearance of the planet, its natural satellite and the satellite's shadow as seen from the Sun (a) and from the Earth (b).

Let us consider the planet's disk appearance as seen from the Sun, i.e. in the heliocentric sky plane (Fig. 1a). The satellite S casts the shadow S_s which cannot be discerned as its position always coincides with the satellite's position. The great-circle arc that bounds the planet's side visible from the Earth, i.e. the planet's light limb projected onto the heliocentric sky plane, is shown with the dotted line.

The numerical method implies that first the transition is made from the satellite's (and its shadow's) coordinates x' and y' projected onto the heliocentric sky plane to non-rotating planetocentric coordinates λ and φ by the known formulae for the oblique orthographic projection (Vakhrameeva, Bugaevskij, Kazakova, 1986). In that coordinate system the longitude λ is reckoned from the planetographic meridian of the subsolar point E westwards of the planet; and the latitude φ is reckoned from the planet's equatorial plane. Then, the planetocentric coordinate system is rotated by the angle ΔA around point E to the subterral point O that specifies the geocentric sky plane orientation. The planet's disk appearance as seen from the Earth, i.e. in the geocentric sky plane, is shown in Fig. 1b. The planetocentric longitude λ_s of the satellite's shadow, which is reckoned from the planet's central meridian passing through point O, is defined by the following formula: $\lambda_s = \lambda + \Delta A$.

And finally, similarly using the known formulae for the oblique orthographic projection, the transition is made from the non-rotating planetocentric coordinates λ_s and φ of the satellite's shadow to its rectangular coordinates x_s, y_s and z_s in the geocentric sky plane.

The transformation of the rates of change in coordinates $\dot{x}_s$, $\dot{y}_s$ and $\dot{z}_s$ is made by a similar way. The numerical method is precise and holds for any phase angle values and any planetocentric declinations of the

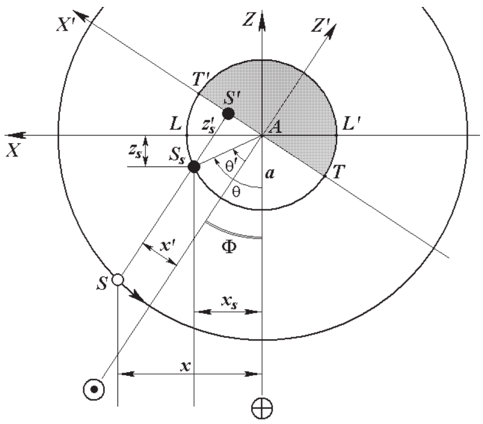

Figure 2: The satellite's shadow position on the planet's surface in its equatorial plane.

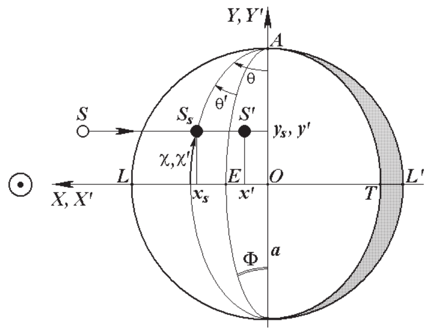

Figure 3: The satellite's shadow on the planet's visible disk in the first approximation.

Earth and the Sun. That allows of its applying for the analytical method testing.

4. The analytical method for determination of the satellite's shadow position and velocity on the illuminated part of the planet's visible disk

Let us consider the problem solution in the first approximation when the Earth and the Sun are assumed to be in the planet's equatorial plane. It means that $D_\odot = 0$ and $D_\oplus = 0$. Hence, it follows that $P - Q = \pm 90°$. In all expressions the sign is selected by the following rule: the upper sign corresponds to the situation when the eastern part of the planet's visible disk is illuminated; and the lower sign implies that its western part is illuminated.

The angles Φ and ΔA are in the planet's equatorial plane, hence, according to formula (1) it can be deemed that $\Phi = \pm\Delta A$.

In a special case of the first approximation when the natural satellite S moves true in the planet's equatorial plane: $y' = 0$, $\dot{y}' = 0$ (Fig. 2), its projection S' onto the heliocentric sky plane changes its position uniformly along the X' axis between points T' and T. The satellite's shadow S_s, which can be observed from the Earth, transits through the planet's equator arc between points L and T. The satellite's shadow position on the planet's surface in its equatorial plane is specified by the angles θ and θ' relative to a geocentric and heliocentric observer, respectively. Hence, $x' = a\sin\theta'$, $z'_s = -a\cos\theta'$ and $x_s = a\sin\theta$, $z_s = -a\cos\theta$ where $\theta = \theta' \pm \Phi$.

In more general case when the satellite's path is parallel to the planet's equatorial plane (Fig. 3), $\dot{y}' = 0$, $x' = a\cos\chi'\sin\theta'$, $y' = a\sin\chi'$, $z' = -a\cos\chi'\cos\theta'$ where the angle χ' is equal to the planetocentric latitude of the satellite's shadow relative to the $X'Y'$ plane. Relative to the XY plane the planetocentric latitude of the satellite's shadow is χ, and $\chi = \chi'$ at that. Hence, $x_s = a\cos\chi\sin\theta$, $y_s = a\sin\chi$ and $z_s = -a\cos\chi\cos\theta$. Those expressions are valid even in the more general case when $\dot{y}' \neq 0$, i.e. the natural satellite moves at an arbitrary angle to the planet's equatorial plane.

Taking into account that the auxiliary coordinate ζ' and its rate of change $\dot{\zeta}'$ are defined by formulae (2) and (3), respectively, the rectangular coordinates of the satellite's shadow are as follows:

$$\left.\begin{aligned} x_s &= x'\cos\Phi \pm a\zeta'\sin\Phi \\ y_s &= y' \\ z_s &= -a\zeta'\cos\Phi \pm x'\sin\Phi \end{aligned}\right\}, \tag{4}$$

and their rates of change are the following:

$$\left.\begin{aligned} \dot{x}_s &= \dot{x}'\cos\Phi \pm a\dot{\zeta}'\sin\Phi \\ \dot{y}_s &= \dot{y}' \\ \dot{z}_s &= -a\dot{\zeta}'\cos\Phi \pm \dot{x}'\sin\Phi \end{aligned}\right\}. \tag{5}$$

Simultaneous formulae (4) and (5) hold for any phase angle values.

Let us consider this case in the second approximation when it is only the Sun is in the planet's equatorial plane ($D_\odot = 0$), and the Earth's planetocentric declination $D_\oplus$ can take on different values (Fig. 4).

The angles Φ and ΔA are situated in different planes; therefore, $\theta = \theta' + \Delta A$, and the values of angles Φ and ΔA are linked by the following expression:

$$\cos\Phi = \cos D_\oplus \cos\Delta A. \tag{6}$$

The sign of angle ΔA in expression (6) is the same as in $\sin(P - Q)$.

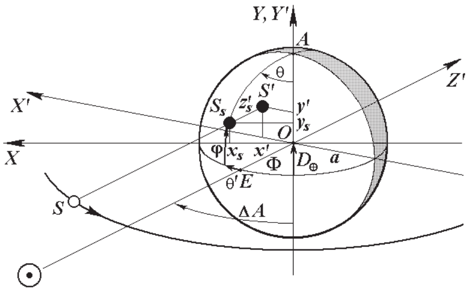

Figure 4: The satellite's shadow on the planet's visible disk in the second approximation.

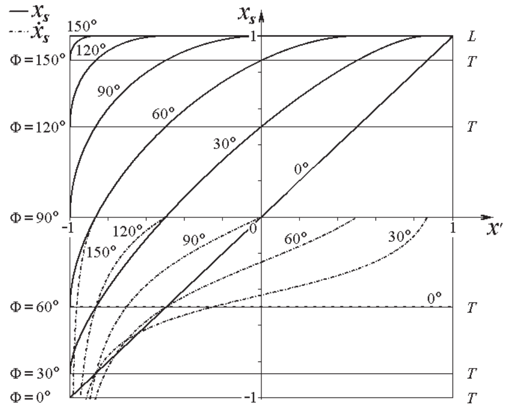

Figure 5: The satellite's shadow coordinate and velocity in a special case in the first approximation.

Therefore, the rectangular coordinates of the satellite's shadow will be equal to

$$\left.\begin{array}{l} x_s = x'\cos\Delta A + a\zeta'\sin\Delta A \\ y_s = y'\cos D_\oplus - (a\zeta'\cos\Delta A - x'\sin\Delta A)\sin D_\oplus \\ z_s = -y'\sin D_\oplus - (a\zeta'\cos\Delta A - x'\sin\Delta A)\cos D_\oplus \end{array}\right\}, \quad (7)$$

and their rates of change are the following:

$$\left.\begin{array}{l} \dot{x}_s = \dot{x}'\cos\Delta A + a\dot{\zeta}'\sin\Delta A \\ \dot{y}_s = \dot{y}'\cos D_\oplus - (a\dot{\zeta}'\cos\Delta A - \dot{x}'\sin\Delta A)\sin D_\oplus \\ \dot{z}_s = -\dot{y}'\sin D_\oplus - (a\dot{\zeta}'\cos\Delta A - \dot{x}'\sin\Delta A)\cos D_\oplus \end{array}\right\}. \quad (8)$$

At the solution of the formulated problem are used only the coordinates x_s and y_s of the satellite's shadow, and also their rate of change $\dot{x}_s$ and $\dot{y}_s$.

The velocity of the satellite's shadow motion across the illuminated part of the planet's visible disk can be defined by the following formula: $V_s = \sqrt{\dot{x}_s^2 + \dot{y}_s^2}$. The simultaneous formulae (7) and (8) hold for any phase angle values and planetocentric declination of the Earth.

If it is deemed that $D_\oplus = 0$ in the second approximation, then it follows from expression (6) that $\Phi = \pm\Delta A$, and the simultaneous formulae (7) and (8) are transformed to the system of formulae (4) and (5).

5. The simulation of the satellite's shadow motion across the illuminated part of the planet's visible disk using the analytical method

The suggested analytical method was tested under various lighting conditions for the planet's visible disk using the numerical method as a reference one. The coordinate z_s of the satellite's shadow and its rate of change $\dot{z}_s$ were used at testing of an analytical method, but for the solution of the formulated problem they have no practical importance. The plot of the x_s coordinate and its rate of change $\dot{x}_s$ against the x' coordinate in a special case in the first approximation at different phase angles are shown in Fig. 5.

As it follows from Fig. 5, the presented dependence is only linear at $\Phi = 0$. At other phase angle values the dependence is nonlinear. As coordinate x' is proportional to time, the satellite's shadow motion across the planet's visible disk is not uniform. The minimum velocity of the satellite's shadow motion is observed near the planet's light limb L, the maximum velocity is observed near the terminator T. With increasing phase angle the nonuniformity of the natural satellite's shadow motion increases.

The analytical method was tested for an example of the Ganymede and its shadow transit across Jupiter's visible disk that occurred on November 12-13, 2002 near the western quadrature (Mikhalchuk, 2007). As per initial data, the instant T'_0 of the satellite's inferior heliocentric conjunction was $19^h30^m.45$ TDT November 12. The daily changes $\dot{x}' = -13.14963$ and $\dot{y}' = -0.00769$ computed at the indicated instant can be deemed constant throughout the transit. The instants T'_1 and T'_2 of the satellite's shadow contact with the illuminated part of the planet's visible disk in the spherical approximation were $17^h44^m.77$ TDT and $21^h18^m.02$ TDT November 12, respectively. An example of computation results at instant $19^h00^m.0$ TDT November 12 is presented in Table 1. For the indicated instant the values of the initial coordinates were $x' = +0.27807$ and $y' = -0.19053$.

The approximations for the numerical method (both the first and the second one) are the same as those for the analytical method. The computation using the precise numerical method was carried out according to the physical ephemerides. As follows from Table 1, the results obtained using different methods (namely, the numerical and analytical ones) are similar in the same approximations; that goes to show the trueness

Table 1: The computation results of the Ganymede's shadow position and velocity on the illuminated part of Jupiter's visible disk obtained with various methods.

Method	Approximation	x_s	y_s	z_s	$\dot{x}_s$	$\dot{y}_s$	$\dot{z}_s$
Numerical	first	+0.44921	−0.19053	−0.87288	−12.19177	−0.00769	−6.27257
Numerical	second	+0.44918	−0.19338	−0.87226	−12.19195	−0.02820	−6.27217
Numerical	precise	+0.44945	−0.18385	−0.87418	−12.19255	+0.01108	−6.27106
Analytical	first	+0.44921	−0.19053	−0.87288	−12.19177	−0.00769	−6.27257
Analytical	second	+0.44918	−0.19338	−0.87226	−12.19195	−0.02820	−6.27217

of the analytical method applied.

6. Conclusion

The key findings of the present study enable to make the following conclusions:

1. The suggested analytical method for determination of the coordinates of the satellite's shadow of the spherical planet on the geocentric plane of the sky and their rate of change allows to receive the solution as the formulae directly connecting these values with coordinates of the satellite on the heliocentric plane of the sky and their rates of change for two approximations of the location of the Earth and the Sun relative to the planet's rotation axis.
2. The formulae obtained in both approximations of the conditions of illumination of the visible planet's disk, are valid for any values of the phase angle and are precise within the limits of the given approximation.
3. The motion of the satellite's shadow on the illuminated part of the visible planet's disk is not uniform, that it is necessary to take into account at the calculation of the ephemerides and at determination of the sizes of the satellite's shadow on the illuminated part of the visible planet's disk.

Acknowledgements. The authors are thankful to anybody who has read this contribution to the end.

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