TWO-BODY PROBLEM IN KALUZA-KLEIN MODELS WITH RICCI-FLAT INTERNAL SPACES

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ABSTRACT. We consider the dynamics of a two-body system in the model with additional spatial dimensions compactified on a Ricci-flat manifold. To define the gravitational field of a system and to construct its Lagrange function we use the weak-field approach. It is shown, that to avoid the contradiction with the experimental restrictions on the value of PPN-parameter γ, the massive sources must have nonzero pressure/tension into the extra dimensions and also must be uniformly smeared there. This fact leads directly to the absence of the Kaluza-Klein modes, which looks unnatural from the point of quantum mechanics.

Key words: Kaluza-Klein models, two-body problem.

1. Introduction

Many modern theories of unification of fundamental interactions, such as superstring theory and its generalizations, are based on the Kaluza-Klein (KK) approach, where the product manifolds with the topology $\mathcal{M}_{\mathcal{D}} = \mathcal{M}_4 \times \mathcal{M}_d$ correspond to the physical spacetime. Here $\mathcal{M}_4$ is the (external) 4-dimensional spacetime and $\mathcal{M}_d$ is a compact (internal) d-dimensional space. The compactification of extra spatial dimensions enables to unify gravity and the Standard Model gauge fields.

The models, based on the KK approach, predict the existence of so-called KK-particles, which correspond to the excited states of the Standard Model particles into the extra dimensions. It is experimentally established, that the lower limit of masses of the KK-particles (and, accordingly, the scale of the internal space) is of the TeV order. Thus, if the scale of extra dimensions exceeds 14 TeV, then it is impossible to find KK-particles and to check the existence of additional dimensions on LHC experiment.

In the light of this fact, it is of interest to investigate the astrophysical consequences of the Kaluza-Klein models, which allows to verify or falsify the KK models with the help of highly accurate gravitational experiments.

Hereinafter, we accept the following notation: Greek indices μ, ν run from 0 to 3, $\tilde{\mu}, \tilde{\nu}$ – from 1 to 3 accordingly, while Latin ones M, N run from 0 to D, $m, n = 4, ..., D$ and $\tilde{M}, \tilde{N} = 1, ... D$, where D is the total dimensionality of the space. The total number of additional spatial dimensions is $d = D - 3$ and $\mathcal{D} = D + 1$. Also X^M is for coordinates on $\mathcal{M}_{\mathcal{D}}$, $x^\mu = X^\mu$ is for coordinates on $\mathcal{M}_4$ and $y^m = X^m$ is for coordinates on M_d.

2. The model

Let

$$[\hat{g}^{(\mathcal{D})}_{MN}(y)] = [\eta^{(4)}_{\mu\nu}] \oplus [\hat{g}^{(d)}_{mn}(y)] \quad (1)$$

be a background metric tensor field on a $\mathcal{D}$-fold $\mathcal{M}_{\mathcal{D}}$. Here $\eta_{00} = 1$, $\eta_{\tilde{\mu}\tilde{\nu}} = -\delta_{\tilde{\mu}\tilde{\nu}}$ and $\hat{g}^{(d)}_{mn}(y)$ is a certain metric tensor on $\mathcal{M}_d$. We perturb this geometric background introducing the matter in the form of two compact gravitating bodies with rest masses m_1 and m_2. We also suppose, that these bodies are pressureless into the external space (this is a natural assumption for such bodies as stars) and generally may have certain pressure p into the compact subspace, which is the immanent property of multidimensional particles. Then the corresponding energy-momentum tensor (EMT) has the following contravariant components

$$\begin{aligned} T^{M\nu} &= \rho c^2 \frac{ds}{dx^0} u^M u^\nu, \quad u^M = \frac{dX^M}{ds}, \\ T^{mn} &= -p g^{mn} + \rho c^2 \frac{ds}{dx^0} u^m u^n. \end{aligned} \quad (2)$$

Here g_{MN} is a *perturbed* metric tensor on $\mathcal{M}_{\mathcal{D}}$, which corresponds to the considering matter distribution, and $ds^2 = g_{MN} dX^M dX^N$. Also ρ is the rest mass D-density of the system. The equation of state into the internal space $p = \omega \rho c^2 (u^0)^{-1}$ contains the parameter of state ω. As we shall see below, ω must be nonzero to provide an accordance with the gravitational experiments.

The dynamics of a system is investigated within the framework of a weak-field approach. To construct the Lagrange function up to $1/c^2$ terms it is necessary to define the perturbed metric components g_{00}, $g_{0\tilde{M}}$ and $g_{\tilde{M}\tilde{N}}$ up to $1/c^4$, $1/c^3$ and $1/c^2$ terms respectfully.

The perturbed metric tensor up to $1/c^2$ terms reads:

$$g_{MN} \approx \hat{g}_{MN} + h_{MN}, \quad g^{MN} \approx \hat{g}^{MN} - h^{MN}, \quad (3)$$

where $h_{MN} \sim O(1/c^2)$ and by definition $h_M^N \equiv \hat{g}^{NT} h_{MT}$. To get these terms we should solve (in corresponding order) the multidimensional Einstein equation:

$$R_{MN} = \frac{2S_D \tilde{G}_{\mathcal{D}}}{c^4} \left[T_{MN} - \frac{1}{D-1} T_L^L g_{MN} \right]. \quad (4)$$

S_D is a total D-dimensional solid angle and $\tilde{G}_{\mathcal{D}}$ is a gravitational constant in $\mathcal{D}$-spacetime.

It's easy to calculate the approximate components and the trace of (1):

$$T_L^L = T_\lambda^\lambda + T_l^l \approx \rho c^2 (1 - \omega d), \quad (5)$$

$$T_{00} \approx \rho c^2, \; T_{\tilde{\mu}\tilde{\nu}} \approx 0, \; T_{mn} \approx -\omega \rho c^2 \hat{g}_{mn}. \quad (6)$$

Hence, for the nonzero (up to $O(1/c^2)$) right-hand sides of (4) we get:

$$R_{00} \approx \frac{2S_D \tilde{G}_{\mathcal{D}}}{c^2} \left(\frac{D - 2 + \omega d}{D-1} \right) \rho, \quad (7)$$

$$R_{\tilde{\mu}\tilde{\nu}} \approx \frac{2S_D \tilde{G}_{\mathcal{D}}}{c^2} \left(\frac{1 - \omega d}{D-1} \right) \rho \, \delta_{\tilde{\mu}\tilde{\nu}}, \quad (8)$$

$$R_{mn} \approx -\frac{2S_D \tilde{G}_{\mathcal{D}}}{c^2} \left(\frac{1 + 2\omega}{D-1} \right) \rho \hat{g}_{mn}, \quad (9)$$

The approximate (up to $1/c^2$ terms) Ricci-tensor components in general case have the following form:

$$R_{MN} \approx \hat{R}_{MN} + \frac{1}{2} \left[-\hat{\nabla}_L \hat{\nabla}^L h_{MN} + Q_{MN} \right], \quad (10)$$

$$\begin{aligned} Q_{MN} \;\; \equiv \;\; & \left[\hat{\nabla}_M \left(\hat{\nabla}_L h_N^L - \frac{1}{2} \partial_N h_L^L \right) \right. \\ & \left. + \hat{\nabla}_N \left(\hat{\nabla}_L h_M^L - \frac{1}{2} \partial_M h_L^L \right) \right] \\ & - \left(\hat{R}^L{}_{NPM} + \hat{R}^L{}_{MPN} \right) h_L^P \\ & + \hat{R}_{PM} h_N^P + \hat{R}_{PN} h_M^P ; \partial_M \equiv \frac{\partial}{\partial X^M} \end{aligned} \quad (11)$$

Hereinafter all symbols marked with "hats" correspond to the background metric $\hat{g}_{MN}$. In particular $\hat{R}^L_{NPM}$ is an unperturbed Riemann tensor and $\hat{\nabla}_M$ is a covariant derivative on the background.

A rather bulky equality (11) may be simplified. Firstly, we use the freedom of coordinate system choice and impose the gauge conditions:

$$\hat{\nabla}_L h_N^L - \frac{1}{2} \partial_N h_L^L = 0. \quad (12)$$

Then a pair of terms in square brackets in (11) is equal to zero. Secondly, if we take into account the isotropic character of the considering EMT (1) with respect to the internal space, we may conclude, that the topology of the internal space remains unchanged after the matter introduction and $[h_{MN}^{(\mathcal{D})}] = [h_{\mu\nu}^{(4)}] \oplus [h_{mn}^{(d)}]$, where $h_{mn}^{(d)} = \xi \hat{g}_{mn}^{(d)}$. The prefactor ξ is a certain scalar field. Taking it into account one can easily find that the rest of terms in (11) vanishes. Consequently, for *the considering matter distribution* (1) and *under the accepted gauge conditions* (12), the relation $R_{MN} \approx \hat{R}_{MN} - (1/2) \hat{\nabla}_L \hat{\nabla}^L h_{MN}$ is valid. In the general case of Ricci-flat internal spaces ($\hat{R}_{mn} = 0$) we obtain up to $1/c^2$-terms:

$$R_{MN} \approx -\frac{1}{2} \hat{\nabla}_L \hat{\nabla}^L h_{MN} \approx -\frac{1}{2} \hat{\nabla}_{\tilde{L}} \hat{\nabla}^{\tilde{L}} h_{MN}. \quad (13)$$

Then the approximate 00-component of the Einstein equations reads:

$$(7), (13) \Rightarrow -\frac{1}{2} \hat{\nabla}_{\tilde{L}} \hat{\nabla}^{\tilde{L}} h_{00} = \frac{2S_D \tilde{G}_{\mathcal{D}}}{c^2} \frac{D - 2 + \omega d}{D-1} \rho \, . \quad (14)$$

Let's suppose that the physically reasonable solution $h_{00}(X^{\tilde{M}})$ of the equation (14) exists. Then it describes the nonrelativistic multidimensional gravitational potential φ of the given matter distribution: $h_{00} \equiv 2\varphi/c^2$. Therefore, using (8), we obtain

$$-\frac{1}{2} \hat{\nabla}_{\tilde{L}} \hat{\nabla}^{\tilde{L}} h_{\tilde{\mu}\tilde{\nu}} = -\frac{1}{2} \hat{\nabla}_{\tilde{L}} \hat{\nabla}^{\tilde{L}} \left(\frac{1 - \omega d}{D - 2 + \omega d} h_{00} \delta_{\tilde{\mu}\tilde{\nu}} \right). \quad (15)$$

Hence

$$h_{\tilde{\mu}\tilde{\nu}} = \frac{1 - \omega d}{D - 2 + \omega d} h_{00} \delta_{\tilde{\mu}\tilde{\nu}}, \quad (16)$$

and, analogically

$$h_{mn} = -\frac{1 + 2\omega}{D - 2 + \omega d} h_{00} \hat{g}_{mn}. \quad (17)$$

These solutions must satisfy the conditions (12). It's not difficult to check, that for the ν-th component of (12) the equality $\hat{\nabla}_L h_\nu^L - (1/2) \partial_\nu h = 0 + O(c^{-3})$ fulfills, while for the n-th component we have the equation:

$$\hat{\nabla}_L h_n^L - \frac{1}{2} \partial_n h = -\frac{\omega(D-1)}{D - 2 + \omega d} \partial_n h_{00} = 0. \quad (18)$$

We consider a general case $\omega \neq 0$, hence, to satisfy (18) we must demand $\partial_n \varphi = 0$. Obviously, as φ depends only on the external coordinates so does ρ. In other words, the massive sources must be uniformly smeared over the internal space: $\rho = \varrho_3 / V_d$, where ϱ_3 is a rest mass density into the external space and $V_d\{g_{mn}^{(d)}\} = \int_{\mathcal{M}_d} d^d y \sqrt{|\det(g_{mn}^{(d)})|}$ is the volume of the internal space.

For two point-like masses, particularly

$$\varrho_3(\mathbf{r}) = [-\det(g_{\mu\nu}^{(4)})]^{-1/2} \sum_{i=1,2} m_i \delta(\mathbf{r} - \mathbf{r}_i), \quad (19)$$

where $\mathbf{r} = (x^1, x^2, x^3)$, and $\mathbf{r}_i$ is a position vector of the i-th particle.

Then the equation (14) is reduced to a 3-dimensional Poisson equation:

$$\triangle_3\varphi(\mathbf{r}) = 4\pi G_N \sum_{i=1,2} m_i\delta(\mathbf{r}-\mathbf{r}_i),\ \triangle_3 \equiv \sum_{\tilde{\mu}} \partial^2_{\tilde{\mu}}. \quad (20)$$

The Newtonian gravitational constant is connected with the multidimensional one as follows:

$$G_N = \frac{S_D(D-2+\omega d)}{2\pi\hat{V}_d(D-1)}\,\tilde{G}_{\mathcal{D}}\,,\ \hat{V}_d \equiv V_d\{\hat{g}^{(d)}_{mn}\}. \quad (21)$$

The solution of (20) is a Newtonian potential:

$$\varphi(\mathbf{r}) = -G_N \sum_{i=1,2} \frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}. \quad (22)$$

As we pointed out above, to construct the Lagrange function of the considering system up to $1/c^2$ terms, we must also define the correction terms $f_{00} \sim O(1/c^4)$ and $f_{0\mu} \sim O(1/c^3)$ in the expansions $g_{00} \approx \hat{g}_{00} + h_{00} + f_{00}$ and $g_{0\mu} \approx f_{0\mu}$. The obtained tensor h_{MN} enable us to construct the corresponding approximate Einstein equations. Within the required accuracy, we have for 00 and $0\tilde{\mu}$ approximate covariant components of the EMT (1) and its trace:

$$T_{00} \approx \frac{\varrho_3 c^2}{\hat{V}_d}\left[1 + \frac{3D-4+\omega d}{D-2+\omega d}\,\frac{\varphi}{c^2} + \frac{v^2}{2c^2}\right], \quad (23)$$

$$T_{0\tilde{\mu}} \approx -\frac{\varrho_3 c}{\hat{V}_d} v^{\tilde{\mu}}, \quad (24)$$

$$T \approx \frac{\varrho_3}{\hat{V}_d}(1-\omega d)\left[c^2 + \varphi\frac{D-\omega d}{D-2+\omega d} - \frac{v^2}{2}\right]. \quad (25)$$

Here $v^{\tilde{\mu}} \equiv c dx^{\tilde{\mu}}/dx^0$, $v^2 \equiv \delta_{\tilde{\mu}\tilde{\nu}}v^{\tilde{\mu}}v^{\tilde{\nu}}$. Obviously, for the particle uniformly smeared over the internal space $v^m = 0$. Hence, the right-hand sides of the approximate 00 and $0\tilde{\mu}$ components of (4) are

$$R_{00} \approx \frac{2S_D\tilde{G}_{\mathcal{D}}}{c^2}\,\frac{\varrho_3}{\hat{V}_d}\left[\frac{D-2+\omega d}{D-1} + \frac{v^2}{c^2}\,\frac{D-\omega d}{2(D-1)} + \frac{\varphi}{c^2}\,\frac{3D-4+\omega d}{D-1}\right], \quad (26)$$

$$R_{0\tilde{\mu}} \approx -\frac{2S_D\tilde{G}_{\mathcal{D}}}{c^3}\,\frac{\varrho_3}{\hat{V}_d}v^{\tilde{\mu}}. \quad (27)$$

The corresponding left-hand sides are

$$R_{00} \approx \frac{1}{c^2}\triangle_3\varphi + \frac{1}{2}\triangle_3\left(f_{00} - \frac{2}{c^4}\,\varphi^2\right) + \frac{2}{c^4}\,\frac{D-1}{D-2+\omega d}\,\varphi\triangle_3\varphi, \quad (28)$$

$$R_{0\tilde{\mu}} \approx \frac{1}{2}\triangle_3 f_{0\tilde{\mu}} + \frac{1}{2c^2}\partial_0\partial_{\tilde{\mu}}\varphi\,. \quad (29)$$

The solutions of the equations (26)-(28) and (27)-(29) are (for more details see ref.):

$$f_{00} = \frac{2\varphi^2}{c^4} + \frac{2G_N^2}{c^4}\sum_p \frac{m_p}{|\mathbf{r}-\mathbf{r}_p|}\sum_{q\neq p}\frac{m_q}{|\mathbf{r}_p-\mathbf{r}_q|} - \frac{D-\Sigma}{D-2+\Sigma}\,\frac{G_N}{c^4}\sum_p \frac{m_p v_p^2}{|\mathbf{r}-\mathbf{r}_p|}\,, \quad (30)$$

$$f_{0\tilde{\mu}} = \frac{G_N}{2c^3}\sum_p \frac{m_p}{|\mathbf{r}-\mathbf{r}_p|}\left[\frac{3D-2-\Sigma}{D-2+\Sigma}\,v_p^{\tilde{\mu}} + n_p^{\tilde{\mu}}(\mathbf{n}_p\mathbf{v}_p)\right], \quad (31)$$

where $n_p^{\tilde{\mu}} \equiv (x^{\tilde{\mu}} - x_p^{\tilde{\mu}})/|\mathbf{r}-\mathbf{r}_p|$ and $(\mathbf{n}_p\mathbf{v}_p) \equiv \sum_{\tilde{\mu}} n_p^{\tilde{\mu}} v_p^{\tilde{\mu}}$.

Thus, now we may construct the Lagrange function of the system. For the i-th body the Lagrange function is defined as follows: $L_i = -m_i c(ds_i/dt) = -m_i c^2\sqrt{g_{00} + 2g_{0\tilde{\mu}}v_i^{\tilde{\mu}}/c + g_{\tilde{\mu}\tilde{\nu}}v_i^{\tilde{\mu}}v_i^{\tilde{\nu}}/c^2}$. With the help of the approximate expressions for g_{00}, $g_{0\tilde{\mu}}$ and $g_{\tilde{\mu}\tilde{\nu}}$, we may calculate L_i. Further, to get the Lagrange function $\mathcal{L}$ of a hole system, we use the relation $\partial\mathcal{L}/\partial\mathbf{r}_i = (\partial L_i/\partial\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}_i}$. After a number of calculations we finally obtain the sought-for approximate Lagrange function $\mathcal{L}$ of the two-body system:

$$\mathcal{L} \approx \sum_{i=1}^{2}\frac{m_i v_i^2}{2} + \sum_{i=1}^{2}\frac{m_i v_i^4}{8c^2} + \frac{G_N m_1 m_2}{|\mathbf{r}_1-\mathbf{r}_2|} - \frac{G_N^2 m_1 m_2(m_1+m_2)}{2c^2|\mathbf{r}_1-\mathbf{r}_2|^2} + \frac{G_N m_1 m_2}{2c^2|\mathbf{r}_1-\mathbf{r}_2|}\left[\frac{D-\omega d}{D-2+\omega d}(v_1^2+v_2^2) - \frac{3D-2-\omega d}{D-2+\omega d}\,(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{n}\mathbf{v}_2)(\mathbf{n}\mathbf{v}_2)\right], \quad (32)$$

where $\mathbf{n} \equiv (\mathbf{r}_1-\mathbf{r}_2)/|\mathbf{r}_1-\mathbf{r}_2|$.

3. Experimental restrictions. Conclusions

It's not difficult to check, that in case $\omega = -1/2$ the expression (32) exactly coincides with the relativistic analog. It is of interest to find the empirical restrictions on the value of the parameter ω, which defines the immanent pressure/tension of the matter into the internal space. Let's assume for this purpose, that there may be a certain deviation $\delta\omega$: $\omega = -1/2 + \delta\omega$.

From the formula (16), rewritten for a solitary particle, we conclude that the parameterized post-Newtonian parameter γ is

$$\gamma = \frac{1-\omega d}{1+(1+\omega)d}. \quad (33)$$

The Shapiro time-delay experiment using the Cassini spacecraft gives: $\gamma = 1 + (2.1 \pm 2.3)\times 10^{-5}$. Hence, for $\delta\omega$ the following limitation is valid

$$|\delta\omega| \leq \frac{d+2}{2d}\times 10^{-5}. \quad (34)$$

From the perihelion shift of Mercury experiments it follows the less strong condition (for details see ref.):

$$|\delta\omega| \le \frac{3(d+2)}{8d} \times 10^{-3}. \qquad (35)$$

Anyway, to achieve the accordance with gravitational tests, we must introduce the internal pressure/tension with the parameter of state $\omega = -1/2 \pm (1/2 + 1/d) \times 10^{-5}$ for usual astrophysical bodies. The interpretation of such strange immanent non-kinetic pressure/tension of the matter is not the only problem of the model. As we have shown in sec. 2, the appearance of such pressure leads directly to the uniform smearing of the masses over the internal space, and, as a consequence, to the absence of the excited KK-states. This fact looks very unnatural from the point of quantum physics. Such situation emerges in all KK-models with Ricci-flat internal manifolds, e.g. in models with Calabi-Yau manifolds, which are widely used in the superstring theory.

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