COSMOLOGY, COSMOMICROPHYSICS AND GRAVITATION

PROPERTIES OF THE GRAVITATIONAL LENS MAPPING IN THE VICINITY OF A CUSP CAUSTIC

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ABSTRACT. We derive approximate formulas for the coordinates and magnification of critical images of a point source in a vicinity of a cusp caustic arising in the gravitational lens mapping. In the lowest (zero-order) approximation, these formulas were obtained in the classical work by Schneider&Weiss (1992) and then studied by a number of authors; first-order corrections in powers of the proximity parameter were treated by Congdon, Keeton and Nordgren. We have shown that the first-order corrections are solely due to the asymmetry of the cusp. We found expressions for the second-order corrections in the case of general lens potential and for an arbitrary position of the source near a symmetric cusp. Applications to a lensing galaxy model represented by a singular isothermal sphere with an external shear γ are studied and the role of the second-order corrections is demonstrated.

Key words: gravitational lensing, analytic theory

1. Introduction

The most interesting effects of gravitational lensing involve caustics of the lens mapping. It is well known that there are only two stable catastrophes of a two-dimensional mapping: cusps and folds (see, e.g., [7]). In a neighborhood of a "regular" caustic point any caustic is a smooth curve formed by the fold points and the smoothness is violated in the cusp points. Approximate relations for image amplifications and positions near the caustic have been derived in classical works [7,8]. These relations represent main contributions of the lensing effects (zero approximation) for a vanishing distance from a point source to the caustic. In order to increase the accuracy of the relations, higher order corrections are needed. The corrections to the solutions of the lens equations and their amplifications near the folds have been found in [1,6] up to the first order and in [3,4] up to the second order (see also [9]). In the neighborhood of the cusp the first order corrections were treated in [5], and in more detail in [2]. Here we study the second order corrections.

2. Approximate solutions near the cusp

The normalized gravitational lens equation maps every point $\mathbf{x}$ of the lens plane (image plane) onto point $\mathbf{y}$ of the source plane [7]:

$$\mathbf{y} = \mathbf{x} - \nabla\Phi(\mathbf{x}), \tag{1}$$

where $\Phi(\mathbf{x})$ is a lens mapping potential. In a general case there are several images $\mathbf{X}^{(l)}(\mathbf{y})$ of the point source at $\mathbf{y}$, which are represented by a solutions of (1); here (l) stands for the number of a separate image. Potential $\Phi(\mathbf{x})$ satisfies equation $\Delta\Phi = 2k$, $k(\mathbf{x})$ being the normalized surface mass density on the line of sight. The amplification coefficient of the point source is $K^{(l)}(\mathbf{y}) = 1/\left|J\left(\mathbf{X}^{(l)}(\mathbf{y})\right)\right|$, where $J(\mathbf{x}) \equiv \left|D(\mathbf{y})/D(\mathbf{x})\right|$ is Jacobian of the lens mapping.

We remind that the critical curves of mapping (1) are determined by equation $J(\mathbf{x}) = 0$. The caustic is the image of the critical curve obtained with this mapping.

The standard approach to study the caustic neighborhoods involves approximations to the potential near point p_{cr} of the critical curve by means of a Taylor polynomial. We use a coordinate system on the source plane such that the abscissa axis is a tangent to the caustic. We assume that point p_{cr} of the critical curve is at the origin of the lens plane; it is mapped onto the origin of the source plane; $|y_2|$ is the distance from the tangent to the caustic, y_1 being a shift along the tangent. In case when point p_{cr} maps onto the cusp point, the axis y_1 is a joint limit of tangents to the two branches of the caustic; this will be referred to as axis of the cusp. We denote $k_0 \equiv k(0)$, $\sigma \equiv 1 - k_0$. Equation (1) near the critical point can be written as follows:

$$\begin{aligned}
y_1 &= 2\sigma x_1 + a_1 x_1^2 - a_2 x_2^2 + 2b_2 x_1 x_2 + c_3 x_1^3 - \\
&- 3c_1 x_1 x_2^2 - d_1 x_2^3 + 3d_2 x_1^2 x_2 + g_1 x_2^4 - \\
&- 6g_2 x_1^2 x_2^2 - 4f_1 x_1 x_2^3 + 4f_2 x_1^3 x_2 + g_3 x_1^4 + \ldots \\
y_2 &= b_2 x_1^2 - b_1 x_2^2 - 2a_2 x_1 x_2 + d_2 x_1^3 - \\
&- 3d_1 x_1 x_2^2 + c_3 x_2^3 - 3c_1 x_2 x_1^2 + f_3 x_2^4 + f_2 x_1^4 - \\
&- 6f_1 x_1^2 x_2^2 + 4g_1 x_1 x_2^3 - 4g_2 x_1^3 x_2 + h x_2^5 + \ldots
\end{aligned} \tag{2}$$

The coefficients of this expansion can be expressed by means of derivatives of $\Phi(\mathbf{x})$. In case of constant surface mass density $k(\mathbf{x}) \equiv k_0$ we have $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$, $g_1 = g_2 = g_3$, $f_1 = f_2 = f_3$.

The condition that $\mathbf{x}=0$ represents a cusp point is $b_1=0$. The parameter t that defines a vicinity of the source to the caustic can be introduces as follows: $y_1=t^2\tilde{y}_1$, $y_2=t^3\tilde{y}_2$, $x_1=t^2\tilde{x}_1$, $x_2=t\tilde{x}_2$. After the substitution into equation (2) we find:

$$\begin{aligned}\tilde{y}_1 &= 2\sigma\tilde{x}_1-a_2\tilde{x}_2{}^2+\left(2b_2\tilde{x}_1\tilde{x}_2-d_1\tilde{x}_2{}^3\right)\cdot t+\\ &+\left(a_1\tilde{x}_1{}^2-3c_1\tilde{x}_1\tilde{x}_2{}^2+g_1\tilde{x}_2{}^4\right)\cdot t^2+\ldots\\ \tilde{y}_2 &= -2a_2\tilde{x}_1\tilde{x}_2+c_2\tilde{x}_2{}^3+\left(b_2\tilde{x}_1^2-3d_1\tilde{x}_1\tilde{x}_2{}^2+\right.\\ &\left.+f_3\tilde{x}_2{}^4\right)\cdot t+\left(-3c_1\tilde{x}_1{}^2\tilde{x}_2+4g_1\tilde{x}_1\tilde{x}_2{}^3+h\tilde{x}_2{}^5\right)\cdot t^2+\ldots\end{aligned} \tag{3}$$

This yields the second approximation (in powers of t) in the neighborhood of the cusp. The lens model parameters are $a_1,a_2,b_2,c_1,c_2,d_1,f_3,g_1,h$; furthermore we shall omit most of the indexes assuming: $a=a_2$, $b=b_2$, $c=c_2$, $d=d_1$, $f=f_3$, $g=g_1$.

We look for the solutions of Eq. (3) in the form: $\tilde{x}_1=x_{10}+tx_{11}+t^2x_{12}$, $\tilde{x}_2=x_{20}+tx_{21}+t^2x_{22}$. We write

$$J(\mathbf{x}(\mathbf{y}))=t^2\left[J_0(\tilde{\mathbf{y}})+tJ_1(\tilde{\mathbf{y}})+t^2J_2(\tilde{\mathbf{y}})\right]. \tag{4}$$

The problem is reduced to the third order algebraic equation for x_{20}:

$$Cx_{20}^3-a\tilde{y}_1x_{20}-\sigma\tilde{y}_2=0\,, \tag{5}$$

where $C=c\sigma-a^2$. Equation (5) has one or three real roots depending on the sign of $Q=\dfrac{\tilde{y}_2^2\sigma^2}{4C^2}-\dfrac{a^3\tilde{y}_1^3}{27C^3}$. We have one real root if $Q>0$ and three real roots if $Q\le 0$ (two of the roots are matched for $Q=0$). On the axis $y_2=0$ we have a trivial root $x_{20}^{(1)}=0$ and two additional real roots $x_{20}^{(2,3)}=\pm\sqrt{a\tilde{y}_1/C}$ for $a\left(\sigma c-a^2\right)y_1>0$. The (real) expressions for the solutions of (5), that is a zero-order approximation for x_{20}, are obtained by means of Cardano's formula.

For the first coordinate we get:

$$x_{10}=\frac{1}{2\sigma}\left(\tilde{y}_1+ax_{20}^2\right). \tag{6}$$

The first order corrections x_{11} and x_{21}, as well as J_1 are found in [2, 5].

3. The second order corrections in case of a locally symmetric cusp

It is an important to note that the first order correction studied in [2, 5] are related solely to an asymmetry of the lens mapping with respect to transformation $y_2\to -y_2$. We shall say that the lens is locally symmetric near the cusp if the following conditions are satisfied (with a sufficient accuracy):

$$y_1(x_1,-x_2)=y_1(x_1,x_2),\quad y_2(x_1,-x_2)=-y_2(x_1,x_2). \tag{7}$$

It is easy to see in virtue of (3) that

$$b=d=f=0 \tag{8}$$

and all terms of the first order in t are absent.

Equation (3) in this case takes on the form:

$$\begin{aligned}\tilde{y}_1 &= 2\sigma\tilde{x}_1-a\tilde{x}_2{}^2+\left(a_1\tilde{x}_1{}^2-3c_1\tilde{x}_1\tilde{x}_2{}^2+g\tilde{x}_2{}^4\right)\cdot t^2+\ldots\\ \tilde{y}_2 &= -2a\tilde{x}_1\tilde{x}_2+c\tilde{x}_2{}^3+\left(-3c_1\tilde{x}_1{}^2\tilde{x}_2+4g\tilde{x}_1\tilde{x}_2{}^3+h\tilde{x}_2{}^5\right)\cdot t^2+\ldots\end{aligned} \tag{9}$$

We seek for the solutions in the form: $\tilde{x}_1=x_{10}+t^2x_{12}$, $\tilde{x}_2=x_{20}+t^2x_{22}$. We obtain:

$$x_{12}=\frac{1}{8\sigma^3C^2E}\left(A_3x_{20}^2\tilde{y}_1^2+A_4x_{20}\tilde{y}_1\tilde{y}_2+A_5\tilde{y}_2^2-C^2aa_1\tilde{y}_1^3\right), \tag{10}$$

$$\begin{aligned}A_3 &= -a^6a_1+3\sigma a^4a_1c-3\sigma^2a^2a_1c^2+\\ &+\sigma^3\left(24a^2cg+8a^3h-18ac^2c_1+3a_1c^3\right),\end{aligned} \tag{11}$$

$$\begin{aligned}A_4 &= \sigma a^5a_1-3\sigma^2a^3a_1c+6\sigma^3\left(2a^3g-3a^2cc_1+\right.\\ &\left.+aa_1c^2\right)+2\sigma^4\left(18acg+8a^2h-9c^2c_1\right),\end{aligned} \tag{12}$$

$$\begin{aligned}A_5 &= -\sigma^2a^4a_1+3\sigma^3a^2a_1c+\\ &+6\sigma^4\left(2a^2g-3acc_1\right)+4\sigma^5\left(3cg+2ah\right).\end{aligned} \tag{13}$$

For the second coordinate we find:

$$x_{22}=\frac{1}{4\sigma C^2E}\left(CB_3x_{20}^2\tilde{y}_2+B_4x_{20}\tilde{y}_1^2+B_5\tilde{y}_1\tilde{y}_2\right), \tag{14}$$

$$B_3=a^3a_1-9\sigma a^2c_1+12\sigma^2a\,g_1+4\sigma^3h\,, \tag{15}$$

$$\begin{aligned}B_4 &= \sigma\left(-6a^2cc_1+aa_1c^2+4a^3g\right)+\\ &+\sigma^2\left(8acg+4a^2h-3c^2c_1\right),\end{aligned} \tag{16}$$

$$\begin{aligned}B_5 &= -a^4a_1+\sigma\left(3a^3c_1+2a^2a_1c\right)+\\ &+4\sigma^2\left(-3acc_1+a^2g\right)+4\sigma^3\left(ah+2cg\right).\end{aligned} \tag{17}$$

For the Jacobian we find:

$$J_2=\frac{1}{2\sigma^2CE}\left(I_3x_{20}^2y_1^2+I_4x_{20}y_1y_2+I_5y_1^3+I_6y_2^2\right), \tag{18}$$

$$\begin{aligned}I_3 &= a^5a_1+\sigma\left(3a^4c_1-2a^3a_1c\right)+3\sigma^2\left(14a^2cc_1-\right.\\ &\left.-aa_1c^2-16a^3g\right)-\sigma^3\left(16a^2h+9c^2c_1\right),\end{aligned} \tag{19}$$

$$\begin{aligned}I_4 &= -\sigma a^4a_1+3\sigma^2\left(3a^3c_1+2a^2a_1c\right)+6\sigma^3\times\\ &\times\left(18acc_1-3a_1c^2-22a^2g\right)-4\sigma^4\left(13ah+6cg\right),\end{aligned} \tag{20}$$

$$I_5=-C\left(a^2a_1+3\sigma ac_1\right), \tag{21}$$

$$\begin{aligned}I_6 &= 9\sigma^2\left[a^3a_1+\sigma\left(3a^2c_1-2aa_1c\right)-\right.\\ &\left.-6\sigma^2\left(2ag-cc_1\right)-4\sigma^3h\right]\end{aligned} \tag{22}$$

Using (4) we get the second approximation formula for the amplification coefficient of a separate image:

$$K=\frac{1}{|J|}\approx\frac{1}{t^2}\frac{1}{\left|J_0+t^2J_2\right|}\approx\frac{1}{t^2|J_0|}\left(1-t^2\frac{J_2}{J_0}\right). \tag{23}$$

In this derivation we have taken into account that for a sufficiently small t we have $\left|t^2J_2/J_0\right|<1$. We denote the first and the second terms of expansion in powers of t in the right hand side of (23) as $K_0\sim t^{-2}$ and $K_2\sim O(1)$. We note that K_2 does not depend on t.

4. Role of the second order corrections

We shall illustrate this role using an example of a simple lens model, namely the singular isothermal sphere with an external shear. The normalized surface mass density of this model is

$$k=\frac{1}{2\sqrt{\left(X_1^2+X_2^2\right)}}\,. \tag{24}$$

The large-scale external gravitational field causes a tidal shear γ [7]. Here (X_1, X_2) stand for coordinates with an origin in a galactic center; the corresponding coordinates in the source plane will be denoted as (Y_1, Y_2).

The lens potential is

$$\Phi(X_1, X_2) = \sqrt{(X_1^2 + X_2^2)} - \frac{\gamma}{2}(X_1^2 - X_2^2), \quad (25)$$

and the equation (1) takes on the form:

$$Y_1 = X_1\left[1 + \gamma - (X_1^2 + X_2^2)^{-1/2}\right], \quad (26)$$

$$Y_2 = X_2\left[1 - \gamma - (X_1^2 + X_2^2)^{-1/2}\right]. \quad (27)$$

It is convenient to represent Jacobian $J(\mathbf{X}) \equiv |D(\mathbf{Y})/D(\mathbf{X})|$ using the polar coordinates

$$J = 1 - \gamma^2 - \frac{1}{R}(1 + \gamma\cos(2\varphi)). \quad (28)$$

The equation of a critical curve $J = 0$ yields:

$$R_{cr}(\varphi, \gamma) = \frac{1 + \gamma\cos(2\varphi)}{1 - \gamma^2}, \quad (29)$$

and substitution of (29) to (26, 27) yields the caustic equation in a parametric form. The critical curves (above) and corresponding caustics are shown on Fig.1 for three values of parameter γ (we assume that $0 < \gamma < 1$).

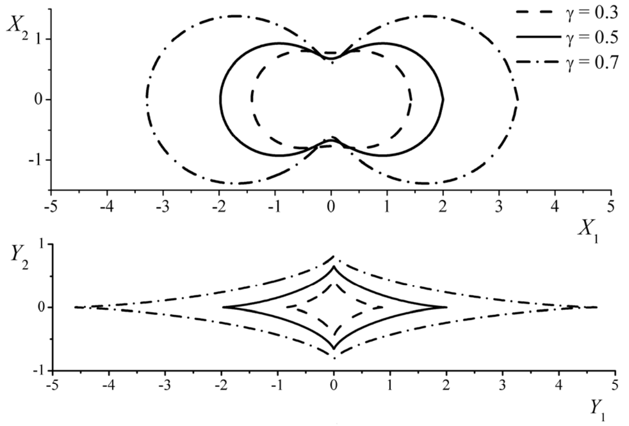

Figure 1: Critical curves and caustics of singular isothermal sphere with an external shear (for three values of γ)

For the cusps we have either $Y_1 = 0$ or $Y_2 = 0$. Consider, for example, a neighborhood of the cusp with coordinates $\mathbf{Y} = \left(\frac{2\gamma}{1-\gamma}, 0\right)$. We shall compare exact solutions with the approximate ones. Let $y_2 = 0$ for the point source. In this case there is a solution of (26, 27) $x_2 = 0$, $x_1 = y_1/(1-\gamma)$, which is defined for $y_1 > -(1+\gamma)/(1-\gamma)$. The corresponding Jacobian is $J = \frac{(1+\gamma)(1-\gamma)^2 y_1}{1 + \gamma + (1-\gamma)y_1}$. In this case the image coordinates are equal to their values obtained in zero approximation. However, this is not so for the Jacobian because $J_0 = (1-\gamma)^2 y_1$; $K_2 = \frac{sign(y_1)}{1-\gamma^2}$. The exact value of the amplification in case of $\gamma = 0.5$ is compared on Fig.2 with zero and second approximations (the latter is the same as the exact value, but this coincidence must be considered as an occasional one). In this case $K_2 \approx 1.33 \cdot sign(y_1)$ and this correction is noticeable even for rather a small distance from the cusp.

For $y_1 < 0$, when the source is inside the caustic, there are two additional solutions of (26, 27), which are symmetric with respect to x_1-axis. Their coordinates are:

$$x_1 = \frac{y_1}{2\gamma}, \quad x_2 = \pm\sqrt{\frac{y_1}{\gamma(\gamma-1)} - \frac{y_1^2}{4\gamma^2}}.$$

The corresponding Jacobian is:

$$J = -2(1-\gamma)^2 y_1 - \frac{(1-\gamma)^3}{2\gamma} y_1^2.$$

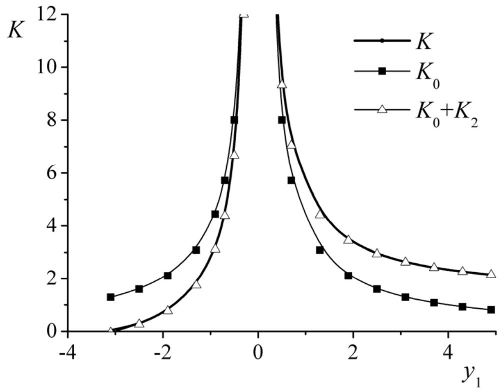

Figure 2: Comparison of the exact (K) and approximate values (K_0 and K_0+K_2) of the amplification

In zero approximation these expressions do not contain the second order terms. Second approximation for J in this case is the same as the exact solution; the amplification is defined by (23), and $K_2 = \frac{1}{8\gamma(1-\gamma)}$.

Fuller consideration of this and other examples have shown convincingly that the second order corrections in the solutions of the lens equation allow us to extend appreciably the domain of validity of the approximation formulae with a preassigned accuracy. Note also that in this approximation we take into account the terms in the amplification coefficient which do not depend upon the vicinity parameter.

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