DETERMINATION OF PARAMETERS OF WHITE DWARF BY THE HYDROGEN SPECTRUM IN THE MODEL WITH LINEAR EQUATION OF STATE

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ABSTRACT. Solutions to Einstein's field equations, for a static spherically symmetric perfect fluid model with linear equation of state are found exactly. It is shown that space of WD can be presented as a space with deformed Heisenberg algebra so WD observational data provide powerful tool in deformed space research. Lane-Emden equation for isothermal model is considered in details.

1. Introduction

We consider in this section the spacetime of non-rotating white dwarf. Exact solutions of General Relativity are hard to come by. A great majority of those known make assumption about symmetry of spacetime, see, for instance, Delgaty & Lake (1998) and Stephani et al. (2003). Only in a few cases, they are presented as general solutions, depending on a few independent continuous parameters such as mass, charge and angular momentum. In general, the non-linearity of partial differential equations of General Relativity makes it difficult to find exact analytical solutions.

Due to the high symmetries of these objects, all non-diagonal elements in the metric vanish, and, due to the static requirements for the gravitational fields, the metric elements are mere functions of the position of a spherically symmetric shell. Static and spherically symmetric non-rotating stars therefore generate a spacetime of the following form

$$ds^2 = -c^2 \exp\left[2\nu(r)\right] dt^2 + \exp\left[2\lambda(r)\right] dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (1)$$

the two functions $\nu(r)$and $\lambda(r)$ are uniquely given by the massenergy distribution $\rho(r)$ in the white dwarf. As in the Newtonian stellar structure, we can define the total mass inside the radius r

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \quad (2)$$

The properties of white dwarf can be obtained by solving the Einstein equations

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{c^4}T_{ab} \quad (3)$$

where R_{ab} and R are Ricci tensor and Ricci scalar respectively. The energy-momentum tensor is given by

$$T^a_b = diag\left(-\rho c^2, P, P, P\right) \quad (4)$$

Equation (3) in the case of white dwarf takes the form

$$\frac{1}{r^2} - \exp(-2\lambda)\left(\frac{1}{r^2} - \frac{2\lambda'}{r}\right) = \frac{8\pi G}{c^2}\rho \quad (5)$$

$$\frac{1}{r^2} - \exp(-2\lambda)\left(\frac{1}{r^2} + \frac{2\nu'}{r}\right) = \frac{-8\pi G}{c^4}P \quad (6)$$

Using the differential energy-momentum conservation law $T^i_{j;i} = 0$ it is easy find out Tolman-Oppenheimer-Volkoff equation

$$\frac{dP}{dr} = \frac{-GM\rho}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi r^3 P}{Mc^2}\right)\left(1-\frac{2GM}{c^2 r}\right)^{-1} \quad (7)$$

Specifying the equation of state is the very first step to find solutions to gravitational field equations for metrics (1). However, despite the growing number of exact static spherically symmetric perfect fluid solutions, most equations of state for known exact solutions have no physical motivation. It seems like these are choosen for specific purpose of simplifying the differential equations, and thereby allowing exact solutions to be found. Nonetheless, simplest models considered in many papers are in accordance (in conformity) with much wider class of models. The solution of the differential equation (Hartle, 1978) for $T = const$ is following

$$\rho = \frac{T}{2\pi G\left(1 + 6\frac{T}{c^2} + \left(\frac{T}{c^2}\right)^2\right)} \cdot \frac{1}{r^2} \quad (8)$$

$$M = \frac{2T}{G\left(1 + 6\frac{T}{c^2} + \left(\frac{T}{c^2}\right)^2\right)} \cdot r \quad (9)$$

and solutions of equations (5) - (6) can be written in the form

$$e^{-2\lambda} = 1 - \frac{4T}{c^2\left(1 + 6\frac{T}{c^2} + \left(\frac{T}{c^2}\right)^2\right)} \quad (10)$$

$$\nu = 2\frac{T}{c^2}e^{2\lambda}\left(1 + \frac{T}{c^2}\right) \cdot \frac{1}{\left(1 + 6\frac{T}{c^2} + \left(\frac{T}{c^2}\right)^2\right)} \cdot \cdot \ln\left(\frac{r}{R}\right) + \nu(R) \quad (11)$$

The line element that describes 4-dimensional homogeneous and isotropic spacetime is given by

$$ds^2 = -c^2 e^{2\nu(R)} \cdot \left(\frac{r}{R}\right)^{\acute{\beta}} dt^2 + e^{2\lambda}dr^2 + + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right) \quad (12)$$

$$\acute{\beta} = 4\frac{T}{c^2}e^{2\lambda} \cdot \left(1 + \frac{T}{c^2}\right) \cdot \frac{1}{\left(1 + 6\frac{T}{c^2} + \left(\frac{T}{c^2}\right)^2\right)} \quad (13)$$

Let introduce new variable z such that

$$z = \frac{Re^{\lambda}}{1 - \frac{\acute{\beta}}{2}}\left(\frac{r}{R}\right)^{1-\frac{\acute{\beta}}{2}} \quad (14)$$

Than line element of spacetime reads

$$ds^2 = \zeta^2\left(\frac{z}{R}\right)^{2\gamma} \cdot \left(-\eta^2 dt^2 + dz^2 + Kz^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) \quad (15)$$

where

$$2\gamma = \frac{\acute{\beta}}{1 - \frac{\acute{\beta}}{2}}$$

$$\zeta^2 = \left[(1 - \frac{\acute{\beta}}{2})e^{-\lambda}\right]^{2\gamma}$$

$$\eta^2 = c^2 e^{2\nu(R)}$$

$$K = \left(1 - \frac{\acute{\beta}}{2}\right)^2 e^{-2\lambda}$$

The line element can be written in terms of flat metric

$$ds^2 = \zeta^2\left(\frac{z}{R}\right)^{2\gamma} \cdot \left(-\eta^2 dt^2 + (1-K)\,dz^2 + Kd\sigma^2\right) \quad (16)$$

where $d\sigma^2$ is the time-independent metric of the 3-dimensional flat space: $d\sigma^2 = \delta_{ij}dz^i dz^j$.

2. Deformed space of White Dwarf

A particle with mass m in space of white dwarf can be described by Dirac-Born-Infeld Lagrangian:

$$L = -\chi \cdot \left(\frac{z}{R}\right)^{\gamma}\sqrt{1 - \frac{K}{\eta^2}\boldsymbol{\upsilon}^2 - \frac{(1-K)}{\eta^2}\upsilon_z^2} \quad (17)$$

here and below a dot denotes the derivative with respect to the time, " $\cdot$ " $\equiv \frac{d}{dt}$ and $|\boldsymbol{\upsilon}| = \frac{d\sigma}{dt}$, $\upsilon_z = \frac{(\mathbf{z},\boldsymbol{\upsilon})}{z} = \dot{z}$. We use the notation $\chi = mc\eta\zeta$. Hamiltonian can be calculated using the usual Legendre transformations

$$H = \frac{\chi\left(\frac{z}{R}\right)^{\gamma}}{1 - \frac{K}{\eta^2}\boldsymbol{\upsilon}^2 - \frac{1-K}{\eta^2}\upsilon_z^2} \quad (18)$$

and can be written in the canonical form

$$H = \sqrt{\chi^2\left(\frac{z}{R}\right)^{2\gamma} + \frac{\eta^2}{K}\mathbf{p}^2 + \frac{(1-K)\,\eta^2}{K}p_z^2} \quad (19)$$

where Hamiltonian is expressed in terms of the momenta $\mathbf{p}$ and $p_z = \frac{(\mathbf{z},\mathbf{p})}{z}$. From expression (14) it can be seen that in $1D$ space we have deformed commutation relation

$$[z, p] = i\beta\left(\frac{z}{R}\right)^{-\gamma} \quad (20)$$

where $\beta = \hbar e^{\lambda}\left[\left(1 - \frac{\acute{\beta}}{2}\right)e^{-\lambda}\right]^{-\gamma}$. A natural generalization of (21) which preserves the rotational symmetry is:

$$[z_i, p_j] = i\beta\left(\frac{z}{R}\right)^{-\gamma}\delta_{ij} \quad (21)$$

In the position representation p_i and z_i act as operators

$$\hat{z}_i\psi(\mathbf{z}) = z_i\psi(\mathbf{z}) \quad (22)$$

$$\hat{p}_i\psi(\mathbf{z}) = -i\beta\left(\frac{z}{R}\right)^{-\gamma}\frac{\partial}{\partial z_i}\psi(\mathbf{z}) \quad (23)$$

3. Coulomb-like problem in space with deformed Heisenberg algebra

In this section we consider Hamiltonian with well-known Coulomb-like potential $U = \frac{-\alpha}{z}$ in space of White Dwarf. From the expression for the Hamiltonian and the representation for z_i and p_i we find out the following generalised form for the stationary state Klein-Gordon equation:

$$E^2\psi + \frac{2E\alpha}{z}\psi + \frac{\alpha^2}{z^2}\psi = \chi^2\left(\frac{z}{R}\right)^{2\gamma}\psi - \frac{\eta^2}{K}\beta^2\left(\frac{z}{R}\right)^{-2\gamma}\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\psi + \frac{\eta^2\beta^2\gamma}{KR^2}\left(\frac{z}{R}\right)^{-2\gamma-2}(\mathbf{z},\nabla)\psi + \frac{(1-K)\eta^2}{K}\frac{(\mathbf{z},\nabla)^2}{z^2}\psi \quad (24)$$

Note that in the spherical variables operator $(\mathbf{z},\nabla)$ act as

$$(\mathbf{z},\nabla)\psi = 3z\frac{\partial}{\partial z}\psi + \tan(\theta)\frac{\partial}{\partial\theta}\psi - 2\cot(2\varphi)\frac{\partial}{\partial\varphi}\psi \quad (25)$$

In order to find the explicit solution it is useful to introduce, as usual, a new variable ξ in terms of which equation (24) takes the form

$$-k_5\left(1-\frac{k_3}{k_5}\xi^{-2\gamma}\right)\frac{\partial^2}{\partial\xi^2}\psi + (2k_3-k_4)\xi^{-2\gamma-1}\frac{\partial}{\partial\xi}\psi - \frac{k_5}{\xi}\frac{\partial}{\partial\xi}\psi + \frac{k_1}{\xi}\psi + \frac{k_2}{\xi^2}\psi - \chi^2\xi^{2\gamma}\psi + E^2\psi = 0 \quad (26)$$

where

$$k_1 = \frac{2E\alpha}{R} \quad k_2 = \frac{\alpha^2}{R^2} \quad k_3 = \frac{\eta^2\beta^2}{KR^2} \quad k_4 = \frac{3\eta^2\beta^2\gamma}{KR^2}$$

$k_5 = \frac{9(1-K)\eta^2}{KR^2}$

4. White Dwarf with linear equation of state

As it can be seen from previous sections in general equation of state can be written as

$$P = T(\rho_0)\rho_0 \quad (27)$$

We consider in this section the simplest case with constant temperature, so we have linear equation of state.For the parametrization $\Gamma = 1$ we now introduce dimensionless variables

$$\rho_0 = \rho_c e^{\theta} \qquad r = a\xi \quad (28)$$

whit $\rho_c = \rho_0(0)$ as the central density and a

$$a = \sqrt{\frac{T}{4\pi G\rho_c}} \quad (29)$$

The hydrostatic equilibrium therefore satisfies the following equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\theta'\right) = -e^{\theta} \quad (30)$$

The mass of the white dwarf can be found as

$$M(\xi) = M_0\xi^2\ |\theta'(\xi)\ | \quad (31)$$

where $M_0 = 4\pi\left(\frac{T}{4\pi G}\right)^{\frac{3}{2}}\frac{1}{\sqrt{\rho_c}}$.

Conclusions

In this paper we described space of White Dwarf via Lagrangian formalism and found expresions that allow us to find hydrogen atom spectra. This paper is not complicated beacuse of it is expected to find graphical presentation for hydrogen spectra and consider relativistic Lane-Emden equation, nonethelles it can be helpful in understanding star evolution and physics of compact objects.

References

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