THE PROPERTIS OF TIDAL FORCES IN THE KERR METRIC

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ABSTRACT. The expression for the tidal forces of the two relativistic protons at a distance of the order of the Compton wavelength near a rotating black hole is found. The analysis shows that the tidal forces are dependent on the plane of incidence and sharply increase with increasing Lorentz factor.

Keywords: General relativity - geodesic deviation: Black hole - Kerr metric: Tidal forces.

1. Introduction

The problem of deviation of geodesic is important in the study of motion of n-interacting particles in strong gravitational fields and in particular, the study of the deformation of the gas and dust clouds in the vicinity of black holes.

When driving two or more closely spaced particles in curved space-time their is a deviation of geodesic lines. General view of the geodesic deviation equation of the n-dimensional Riemannian manifold was obtained by T. Levi-Civita in 1925 Ref. 1. For the 4-dimensional space deviation equation for structureless massless particles was investigated in Refs. 2–3 by J.L. Sing.

The system of units $G = c = 1$ is used in the paper.

2. The Kerr Metric

The Kerr's metric in Boyer-Lindquist coordinates has the form [4]:

$$ds^2 = \rho^2 \frac{\Delta}{\Sigma^2} dt^2 - \frac{\Sigma^2}{\rho^2} \left[d\varphi - \frac{2aMr}{\Sigma^2} dt \right]^2 sin^2\theta - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \,, \quad (1)$$

where

$$\Delta = r^2 - 2Mr + a^2 \,, \quad (2)$$

$$\rho^2 = r^2 + a^2 cos^2\theta \,, \quad (3)$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta sin^2\theta \,, \quad (4)$$

and M is the black hole mass, aM its angular momentum $0 \le a \le 1$. The event horizon of the Kerr's black hole corresponds to the coordinate:

$$r_h = M + \sqrt{M^2 - a^2}. \quad (5)$$

The static limit surface is defined by the value:

$$r_{st} = M + \sqrt{M^2 - a^2 cos^2\theta} \,. \quad (6)$$

The region of space-time between the static limit and the event horizon is called ergosphere.

In view of the equation (1) metric tensors are Ref.5:

$$g^{ij} = \begin{pmatrix} \Sigma^2/\rho^2\Delta & 0 & 0 & 2aMr/\rho^2\Delta \\ 0 & -\Delta/\rho^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ 2aMr/\rho^2\Delta & 0 & 0 & -(\Delta - a^2 sin^2\theta)/\rho^2\Delta sin^2\theta \end{pmatrix}. \quad (7)$$

The nonzero components of the curvature tensor in the Kerr metric have the form [5]:

$$R_{1023} = -aMcos\theta(3r^2 - a^2 cos^2\theta)\frac{1}{\rho^6} \,, \quad (8)$$

$$R_{1230} = -\frac{aMcos\theta}{\rho^6}(3r^2 - a^2 cos^2\theta)\Sigma^{-2} \times [(r^2 + a^2)^2 + 2a^2\Delta sin^2\theta] \,, \quad (9)$$

$$R_{1302} = \frac{aMcos\theta}{\rho^6}(3r^2 - a^2 cos^2\theta)\Sigma^{-2} \times [2(r^2 + a^2)^2 + a^2\Delta sin^2\theta] \,, \quad (10)$$

$$-R_{3002} = R_{1213} = -\frac{aMcos\theta}{\rho^6}(3r^2 - a^2 cos^2\theta) \times \frac{3a\Delta^{1/2}}{\Sigma^2}(r^2 + a^2)sin\theta \,, \quad (11)$$

$$-R_{1220} = R_{1330} = -\frac{Mr}{\rho^6}(r^2 - 3a^2 cos^2\theta) \times \frac{3a\Delta^{1/2}}{\Sigma^2}(r^2 + a^2)sin\theta \,, \quad (12)$$
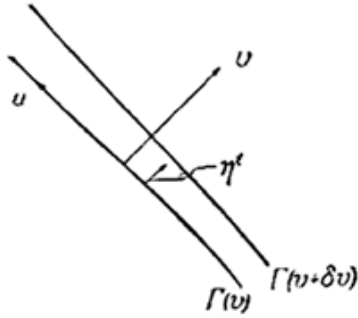
Figure 1: The deviation of curve $\Gamma(v)$ from curve $\Gamma(v + \delta v)$.

$$-R_{1010} = R_{2323} = \frac{Mr}{\rho^6}(r^2 - 3a^2 cos^2\theta)$$
$$= R_{0202} + R_{0303}\,, \qquad (13)$$

$$-R_{1313} = R_{0202} = \frac{Mr}{\rho^6}(r^2 - 3a^2 cos^2\theta)\Sigma^{-2}$$
$$\times [2(r^2 + a^2)^2 + a^2\Delta sin^2\theta]\,, \qquad (14)$$

$$-R_{1212} = R_{0303} = -\frac{Mr}{\rho^6}(r^2 - 3a^2 cos^2\theta)\Sigma^{-2}$$
$$\times [(r^2 + a^2)^2 + 2a^2\Delta sin^2\theta]\,. \qquad (15)$$

3. The equations of geodesic deviation

If there is a pair of adjacent curves $\Gamma(v)$ and $\Gamma(v+\delta v)$ (Fig.1), then the equation of geodesic deviation for structureless infinitely close particles has the form:

$$\frac{d^2\eta^i}{ds^2} + R^i_{jkm}U^j\eta^kU^m = 0\,, \qquad (16)$$

where η^i — infinitesimal vector deviation, $R^i_{jkm} = g^{il}R_{ljkm}$ — Riemann tensor, U^m— 4-speed.

A solution of equation (16) is a vector of deviation of world lines that covariantly describes the relative acceleration between geodesic lines.

Let us consider the equations of geodesic deviation in the Kerr metric. We find the equation of geodesic deviation for relativistic structureless particles that have only radial velocity component, hence:

$$U^i = \Gamma(1, V, 0, 0) \qquad (17)$$

where $\Gamma = \frac{1}{\sqrt{1-V^2}}$ is Lorentz factor.

From the equations (16) and curvature tensor in the Kerr metric (8)—(15) in this case, the equation of the deviation will have the form:

$$\frac{D^2\eta^0}{ds^2} = \Gamma^2 \left[g^{03}(R_{3002} - R_{0303}) + g^{03}V(R_{1230} + R_{1330} \right.$$
$$\left. - R_{1302}) + V^2(g^{00}R_{1010} - g^{03}R_{1313}) \right]\,, \quad (18)$$

$$\frac{D^2\eta^1}{ds^2} = -g^{11}\Gamma^2 R_{1010}(1 + V)\,, \qquad (19)$$

$$\frac{D^2\eta^2}{ds^2} = g^{22}\Gamma^2 \left[R_{0202} + V(R_{1220} + R_{1302}) - R_{1212}V^2 \right]\,, \qquad (20)$$

$$\frac{D^2\eta^3}{ds^2} = \Gamma^2 \left[-g^{33}R_{0303} + g^{33}V(R_{1230} + R_{1330} - R_{1302}) \right.$$
$$\left. + V^2(g^{03}R_{1010} - g^{33}R_{1313}) \right]\,. \qquad (21)$$

From these equations it is seen that the relative acceleration between the infinitely close to the world lines will be directly proportional to the square of the Lorentz factor.

To evaluate the tidal forces of the proton in the Kerr metric we use the following restrictions:

- let proton with mass $m_p = 1.67 \cdot 10^{-27}$kg is in motion along a geodesic so that the deviation is proportional to the Compton wavelength $\lambda^p_C = 1.32 \cdot 10^{-15}$m,

- let movement only occurs in the equatorial plane $\theta = \frac{\pi}{2}$,

- the black hole has the following parameters: $M = 10^6$, $a = 0.98$,

- motion occurs at a coordinate of distance of r from horizon of black hole $r = 10^{-5}r_h$,

- the proton velocity is $V = (1 - 10^{-15})c$.

If we consider the assumptions given above, then calculations made by us lead to the work of the tidal forces $F = 4.156 \cdot 10^9$ Newton's. For example, the same force on the surface of the Sun is about $4.5 \cdot 10^{-26}$ Newtons.

For different values of velocities of particles obtain the tidal forces by numerical calculations for protons are given in the Fig. 2. The graph shows that the tidal forces increase with the speed of the proton in the center of mass of the order of 10^{-19} Newtons with $V = 0.9c$ up to 10^{19}Newtons with $V = (1 - 10^{20})c$.

Dependence of tidal forces on the mass and the specific angular momentum of the black hole is shown in Fig. 3 and Fig. 4 respectively. It is important to note that tidal forces are maximal for black holes of stellar mass and are minimal for a supermassive black holes. For example, if the black hole mass is of about mass of Sun then it creates tidal forces near horizon order 10^{15} Newtons and supermassive black holes with mass $10^9 M_\odot$ creates tidal forces order 10^6 Newtons. A similar effect was observed in Ref. 6 for a Schwarzschild
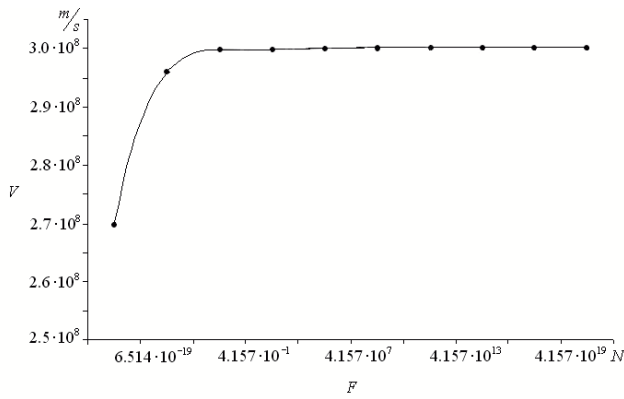
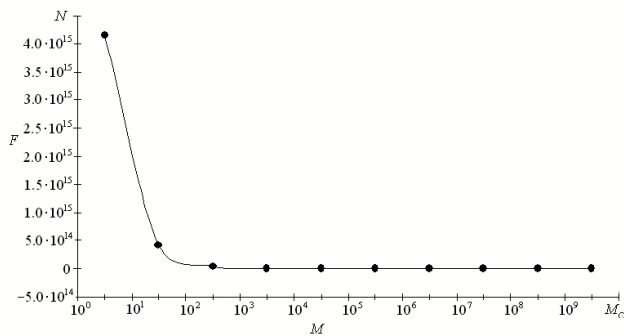Figure 2: The dependence of the tidal force F on the speed of the proton V.

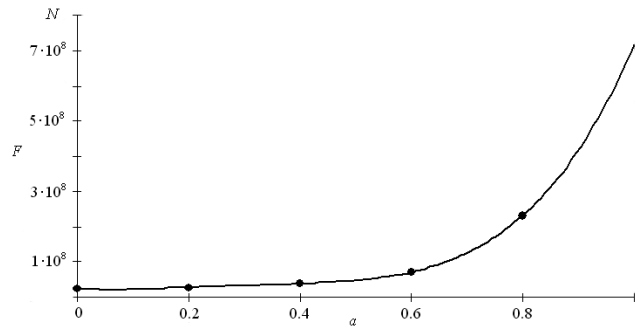Figure 3: The dependence of the tidal force F on the black hole mass M.

Figure 4: The dependence of the tidal force F on the specific angular momentum of the black hole a.

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black hole. Also if there is an increase of specific angular momentum of the black hole then there is an increase of the tidal forces near the horizon.