FIVE DIMENSIONAL BOOSTS AND ELECTROMAGNETIC FIELD IN KALUZA-KLEIN THEORY

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ABSTRACT. A stationary, spherically symmetric, 5D Kaluza-Klein theory exhibits 5D boosts. After reduction of 5D vacuum Einstein action of a diagonal 5D metric, we obtained Einstein equations with energy-momentum tensor of a massless scalar field. Applying a 5D boost generates a non-diagonal metric with an electric field. This electric field is trivial, since it can be removed by a reverse 5D boost. Existence of such kind of trivial fields is analyzed on the example previously known solutions. Transformation properties of physical fields relative to 5D boosts examined. This symmetry is a subgroup of $SL(3,R)$ symmetries of 5D equations, rewritten in $3+2$-decomposition in stationary, spherically symmetric case. This symmetry also can be used to generate new solutions. The solution with trivial electric field is obtained from diagonal 5D metric by a simple symmetry transformation, which reduces to the coordinate 5D boosts.

Keywords: Kaluza-Klein theory, decomposition, electric field, symmetry.

1. Introduction

In a framework of the usual 5D Kaluza-Klein theory (KK), we consider M^5 space with the following 5D metric:

$$^{(5)}ds^2 = {}^{(5)}g_{AB}dx^A dx^B, \tag{1}$$

where $\{A, B = 0,1,2,3,4\} = \{\mu, \nu = 0,1,2,3\} \cup \{4\}$, x^μ –are space-time coordinates, $x^4 = z$ – fifth coordinate.

As it is known, KK theory is based on two postulates:

1. Cylindrical condition, according to which, the space M^5 admits a space-like Killing vector $\vec{\xi}$. In the corresponding coordinate system it has the form $\vec{\xi} = \partial/\partial z$, that leads to metric's independence of the fifth coordinate z, that is $^{(5)}g_{AB} = {}^{(5)}g_{AB}(x^\mu)$.

2. Closure condition, which states that the space M^5 is closed relative to coordinate z.

Four dimensional physical space can be derived by dimensional reduction of M^5 space, and it's corresponding action. This can be accomplished by orthogonal $4+1$ splitting of M^5 space, and then projecting all quantities on the physical space-time M^4. In result we get the following form of the 5D metric:

$$\begin{aligned} ^{(5)}ds^2 &= {}^{(5)}g_{\mu\nu}dx^\mu dx^\nu + 2\,{}^{(5)}g_{\mu 4}dx^\mu dx^4 + {}^{(5)}g_{44}dz^2 \\ &= V^{-1}h_{\mu\nu}dx^\mu dx^\nu - V^2\left(dz + A_\mu dx^\mu\right)^2, \end{aligned} \tag{2}$$

where:

$$^{(5)}g_{\mu\nu} = V^{-1}h_{\mu\nu} - V^2 A_\mu A_\nu\,, \tag{3}$$

$$^{(5)}g_{4\nu} = -V^2 A_\nu, \quad {}^{(5)}g_{44} = -V^2\,. \tag{4}$$

This splitting leads to the following physical space-time metric:

$$h_{\mu\nu} = {}^{(5)}g_{\mu\nu} - \frac{^{(5)}g_{4\nu}\,{}^{(5)}g_{4\mu}}{^{(5)}g_{44}}\,, \tag{5}$$

and the following electromagnetic field:

$$A_\nu = \frac{^{(5)}g_{4\nu}}{^{(5)}g_{44}}\,, \tag{6}$$

while the scalar field is:

$$V^2 = -\left(\vec{\xi}, \vec{\xi}\right) = -{}^{(5)}g_{44}\,. \tag{7}$$

After orthogonalization of the 5D action

$$S = -\frac{1}{4\pi L}\int d^5x\sqrt{^{(5)}g}\ {}^{(5)}R \tag{8}$$

we extract a full derivative, and then integrating by the periodic coordinate z we get:

$$\begin{aligned} S &= -\frac{1}{4\pi L}\int d^4x\sqrt{h}\left\{^{(4)}R - \frac{1}{2}\frac{(\nabla V)^2}{V^2} + \right. \\ &+ \left.\frac{V^3}{4}F_{\mu\nu}F^{\mu\nu}\right\}. \end{aligned} \tag{9}$$

Using the following natural variable substitution:

$$V = e^{\varphi/\sqrt{3}} \tag{10}$$

we get the following 5D metric:

$$
\begin{aligned}
{}^{(5)}ds^2 &= e^{-\varphi/\sqrt{3}} h_{\mu\nu} dx^\mu dx^\nu - \\
&- e^{2\varphi/\sqrt{3}} \left(dz + A_\mu dx^\mu\right)^2 \qquad (11)
\end{aligned}
$$

and following 4D action of Einstein's form:

$$
\begin{aligned}
S &= -\frac{1}{4\pi L} \int d^4x \sqrt{h} \left\{ {}^{(4)}R - \frac{1}{2} \left(\nabla\varphi\right)^2 + \right. \\
&+ \left. \frac{1}{4} e^{\sqrt{3}\varphi} F_{\mu\nu} F^{\mu\nu} \right\} . \qquad (12)
\end{aligned}
$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is an electromagnetic tensor, φ is a massless scalar field.

In summary, 5D variational principle for 5D manifold with a metric that subject to cylindrical and closure conditions, is equivalent to a 4D variation principle of a system with an interacting scalar, electromagnetic and gravitational fields, and above mentioned action.

2. Stationary space of 5D theory and boosts

Suppose that on M^5 space, in addition to the space-like Killing vector $\vec{\xi}_4$, there is also a time-like Killing vector $\vec{\xi}_0 = \partial/\partial x^0$ orthogonal to space directions. Thus we have

$$
ortgonal \vec{\xi}_a = \frac{\partial}{\partial x^a} \qquad (a = 0, 4). \qquad (13)
$$

Those two vectors induce a "2+3 splitting" of M^5 space, and the metric becomes:

$$
\begin{aligned}
{}^{(5)}ds^2 &= {}^{(2)}ds^2 + {}^{(3)}ds^2 \\
&= \gamma_{ab}\left(x^k\right) dx^a dx^b - h_{ij}\left(x^k\right) dx^i dx^j , (14)
\end{aligned}
$$

where $\{a, b = 0, 4\}$ and $\{i, j, k = 1, 2, 3\}$. The Killing vector $\vec{\xi}_a$ is subject to the transformations:

$$
\vec{\eta}_{\tilde{a}} = L^b_{\tilde{a}} \vec{\xi}_b = L^b_{\tilde{a}} \frac{\partial}{\partial x^b} = \frac{\partial}{\partial x^{\tilde{a}}} , \qquad (15)
$$

where $\vec{\eta}_{\tilde{a}}$ – new Killing vectors, $L^b_{\tilde{a}}$ – a constant invertible matrix. Herewith, this "2+3 splitting" of the metric (14) is form invariant. This transformation, can be also induced by the following linear coordinate transformation $\{x^0, x^4\} = \{x^a\}$:

$$
\begin{gathered}
\tilde{x}^{\tilde{a}} = L^{\tilde{a}}_b x^b , \quad L^b_{\tilde{a}} L^{\tilde{a}}_c = \delta^b_c , \\
\det \left\| L^b_a \right\| = L^0_0 L^4_4 - L^0_4 L^4_0 = \pm 1 .
\end{gathered}
$$

In this text we will use only the positive sign. This transformations keeps the interval ${}^{(2)}ds^3$ structure unchanged. While the 2D metric $\gamma_{ab}(x^k)$ transforms according to:

$$
\tilde{\gamma}_{\tilde{a}\tilde{b}}\left(\tilde{x}^k\right) = L^b_{\tilde{a}} L^b_{\tilde{b}} \gamma_{ab}\left(x^k\right) .
$$

Let the metrics $\tilde{\gamma}_{\tilde{a}\tilde{b}}\left(\tilde{x}^k\right)$ and $\gamma_{ab}\left(x^k\right)$ are asymptotically flat, i.e. in spatial infinity our metric metric is pseudo-euclidean:

$$
\lim_{r\to\infty} \gamma_{ab}\left(x^k\right) = \eta_{ab}, \; \lim_{r\to\infty} \tilde{\gamma}_{\tilde{a}\tilde{b}}\left(\tilde{x}^k\right) = \eta_{\tilde{a}\tilde{b}} ,
$$

where η_{ab} – is 2D Minkowski tensor. Then:

$$
\begin{aligned}
\eta_{\tilde{a}\tilde{b}} &= \lim_{r\to\infty} \tilde{\gamma}_{\tilde{a}\tilde{b}}\left(\tilde{x}^k\right) = \lim_{r\to\infty} L^a_{\tilde{a}} L^b_{\tilde{b}} \gamma_{ab}\left(x^k\right) \\
&= L^a_{\tilde{a}} L^b_{\tilde{b}} \lim_{r\to\infty} \gamma_{ab}\left(x^k\right) = L^a_{\tilde{a}} L^b_{\tilde{b}} \eta_{ab} .
\end{aligned}
$$

Thus, those transformations, at infinity approaches 2D Lorentz transformations, or 5D boosts in $\{x^a\} = \left\{x^0 = t, \; x^4 = z\right\}$ coordinates, with a property:

$$
L^a_{\tilde{a}} L^b_{\tilde{b}} \eta_{ab} = \eta_{\tilde{a}\tilde{b}} . \qquad (16)
$$

In consequence, we have 2D Lorentz transformations:

$$
\begin{aligned}
x^0 &= \frac{\tilde{x}^0 + v\tilde{x}^4/c}{\sqrt{1 - V^2/c^2}} = \tilde{x}^0 \cosh\alpha + \tilde{x}^4 \sinh\alpha , \\
x^4 &= \frac{\tilde{x}^4 + v\tilde{x}^0/c}{\sqrt{1 - V^2/c^2}} = \tilde{x}^0 \sinh\alpha + \tilde{x}^4 \cosh\alpha ,
\end{aligned}
$$

where:

$$
\tanh\alpha = \frac{v}{c} ,
$$

Herewith, the physical quantities transforms as:

$$
\tilde{h}_{\tilde{0}\tilde{0}} = -\frac{\det\gamma_{ab}}{\sqrt{-V^{-1} h_{00} \sinh^2\alpha + V^2 \left(A_0 \sinh\alpha + \cosh\alpha\right)^2}} ;
$$

$$
\tilde{A}_{\tilde{0}} = \frac{\left(V^{-3} h_{00} - A_0^2 - 1\right) \tanh\alpha - A_0 \left(1 + \tanh^2\alpha\right)}{V^{-3} h_{00} \tanh^2\alpha - \left(A_0 \tanh\alpha + 1\right)^2} ;
$$

$$
\tilde{V}^2 = -V^{-1} h_{00} \sinh^2\alpha + V^2 \left(A_0 \sinh\alpha + \cosh\alpha\right)^2 .
$$

Note that the metric' determinant

$$
\det\gamma_{ab} = \gamma_{00}\gamma_{44} - \left(\gamma_{04}\right)^2 = -\frac{h_{00}}{V} = inv \qquad (17)
$$

actually an invariant.

Those transformations, physically corresponds to using a moving frame of reference of the fifth coordinate $\tilde{x}^4$, which according to the Kaluza-Klein postulates, is closed.

Let our initial metric be diagonal, this will correspond to that there is no electromagnetic field. Owing to 5D boosts, and bearing in mind that $\gamma_{04} = 0$, and $A_0 = \gamma_{04}/\gamma_{44} = 0$ we get:

$$
\begin{aligned}
\tilde{h}_{\tilde{0}\tilde{0}} &= \frac{-V h_{00}}{\sqrt{V^2 \cosh^2\alpha - V^{-1} h_{00} \sinh^2\alpha}} , \qquad (18) \\
\tilde{A}_{\tilde{0}} &= \frac{\left(V^{-1} h_{00} - V^2\right) \tanh\alpha}{V^{-1} h_{00} \tanh^2\alpha - V^2} , \qquad (19) \\
\tilde{V}^2 &= V^2 \cosh^2\alpha - V^{-1} h_{00} \sinh^2\alpha . \qquad (20)
\end{aligned}
$$

So we see that after 5D Lorentz transformations, the metric becomes non-diagonal. In 4D physical space-time it will correspond the appearance of the electromagnetic field. This means that we actually generating electromagnetic field with its corespondent effective electric charge. This electric field corresponds to a global 5D boost, so we call it trivial field, due to the fact that applying backward 5D boost (by coordinate transformation $\tilde{x}^{\tilde{a}} = L_b^{\tilde{a}} x^b$) this field can be removed (a removable electric charge).

To find the condition under which this electromagnetic field can be removed, we consider the transformation of the metric ${}^{(2)}ds^2$ by the 2D boost $x^b = L_{\tilde{a}}^b \tilde{x}^{\tilde{a}}$:

$$\begin{aligned} \tilde{\gamma}_{\tilde{a}\tilde{b}}\left(\tilde{x}^k\right) &= L_{\tilde{a}}^a L_{\tilde{b}}^b \gamma_{ab}\left(x^k\right), && (21)\\ \det\left(\tilde{\gamma}_{\tilde{a}\tilde{b}}\right) &= \det\gamma_{ab} = inv\,. && (22)\end{aligned}$$

From this we have:

$$\begin{aligned} \tilde{\gamma}_{\tilde{0}\tilde{0}} &= \gamma_{00}\cosh^2\alpha + \gamma_{04}\sinh 2\alpha + \gamma_{44}\sinh^2\alpha && (23)\\ \tilde{\gamma}_{\tilde{0}\tilde{4}} &= \frac{1}{2}\left(\gamma_{00}+\gamma_{44}\right)\sinh 2\alpha + \gamma_{04}\cosh 2\alpha\,, && (24)\\ \tilde{\gamma}_{\tilde{4}\tilde{4}} &= \gamma_{00}\sinh^2\alpha + \gamma_{04}\sinh 2\alpha + \gamma_{44}\cosh^2\alpha && (25)\end{aligned}$$

The requirement $\tilde{\gamma}_{\tilde{0}\tilde{4}} = 0$ leads us to the needed condition for 5D metric to be diagonalizable:

$$\frac{\gamma_{04}\left(x^k\right)}{\gamma_{00}\left(x^k\right)+\gamma_{44}\left(x^k\right)} = -\frac{1}{2}\tanh 2\alpha = const\,. \qquad (26)$$

In terms of physical quantities, this conditions takes the form:

$$\frac{V^{-3}h_{00} - A_0^2 - 1}{A_0} = \coth\alpha + \tanh\alpha = const\,. \qquad (27)$$

While the rest of the metric components will transform as:

$$\begin{aligned} \tilde{\gamma}_{\tilde{0}\tilde{0}} &= \frac{\gamma_{00}\cosh^2\alpha - \gamma_{44}\sinh^2\alpha}{\cosh 2\alpha}\,, && (28)\\ \tilde{\gamma}_{\tilde{4}\tilde{4}} &= \frac{\gamma_{44}\cosh^2\alpha - \gamma_{00}\sinh^2\alpha}{\cosh 2\alpha}\,. && (29)\end{aligned}$$

Analyzing the conditions that will keep fifth coordinate space-like (i.e. conserve metric signature) when electric field is generated, with a consideration of the initial values $\gamma_{04} = 0$, and $A_0 = \gamma_{04}/\gamma_{44} = 0$, we have:

$$\tilde{V}^2 = V^2\cosh^2\alpha - V^{-1}h_{00}\sinh^2\alpha \geq 0$$

$$\frac{v^2}{c^2} = \tanh^2\alpha \leq \frac{V^3}{h_{00}} = \frac{\left(\gamma_{44}\right)^2}{-\det\gamma_{ab}} = \frac{-\gamma_{44}}{\gamma_{00}}\,.$$

Thus, the allowed global boosts is when the coordinate velocity not larger than:

$$v \leq c\sqrt{-\frac{\gamma_{44}}{\gamma_{00}}}\,. \qquad (30)$$

For asymptotically flat space, we conclude the expected result of $v \leq c$. With higher speeds, horizons are formed, and the meaning of fifth coordinate does change.

3. Examples of a removable and unremovable electric charge

3.1. A removable electric charge

In Chodos and Detweiler (1982) work, they obtained the following solution:

$$\begin{aligned} {}^{(5)}ds^2 &= -e^{\mu}dt^2 + 2\tilde{A}dtdx + \varphi^2 dx^2 + \\ &\quad + e^{\beta}\left[dr^2 + r^2 d\sigma^2\right] && (31)\\ &= {}^{(4)}ds^2 + \varphi^2\left(dx + Adt\right)^2\,, \\ {}^{(4)}ds^2 &= -e^{\nu}dt^2 + e^{\beta}\left[dr^2 + r^2 d\sigma^2\right] && (32)\\ A &= \frac{\tilde{A}}{\varphi^2},\ e^{\nu} = e^{\mu} + \frac{\tilde{A}^2}{\varphi^2} && (33)\end{aligned}$$

where

$$\begin{aligned} \varphi^2 &= a_1\psi^{p_1} + a_2\psi^{p_2},\ \psi = \left(\frac{r-B}{r+B}\right)^{\lambda/2B}, && (34)\\ e^{\nu} &= e^{\beta}e^{\mu} + \frac{\tilde{A}}{\varphi^2} = \frac{\psi^2}{\varphi^2}\,, && (35)\\ e^{\beta} &= \left(1 - \frac{B^2}{r^2}\right)^2 \frac{1}{\psi^2}\,, && (36)\\ \tilde{A} &= \sqrt{-a_1a_2}\left(\psi^{p_1} - \psi^{p_2}\right)\,, && (37)\\ e^{\mu} &= a_2\psi^{p_1} + a_1\psi^{p_2}\,, && (38)\\ A &= \frac{\tilde{A}}{\varphi^2} = \frac{\sqrt{-a_1a_2}\left(\psi^{p_1} - \psi^{p_2}\right)}{a_1\psi^{p_1} + a_2\psi^{p_2}}\,, && (39)\\ E &\equiv F_{rt}e^{-\nu/2} = \frac{Q}{r^2\varphi^3}e^{-\beta/2}\,, && (40)\\ p_{1,2} &= 1 \pm \sqrt{1+\kappa},\ \kappa = 4\left(\frac{4B^2}{\lambda^2} - 1\right), && (41)\\ Q^2 &= -a_1a_2\left(1+\kappa\right)\frac{\lambda^2}{G}\,, \quad a_1 + a_2 = 1\,, && (42)\end{aligned}$$

where a_1, a_2, p_1, p_2, λ, and $B-$ constants.

We rewrite this metric in the form:

$$\begin{aligned} {}^{(5)}ds^2 &= -\left(a_2\psi^{p_1} + a_1\psi^{p_2}\right)dt^2 + \\ &\quad + 2\left(\sqrt{-a_1a_2}\left(\psi^{p_1} - \psi^{p_2}\right)\right)dtdx + \\ &\quad + \left(a_1\psi^{p_1} + a_2\psi^{p_2}\right)dx^2 + e^{\beta}\left[dr^2 + r^2 d\sigma^2\right]\end{aligned}$$

or:

$$\begin{aligned} {}^{(5)}ds^2 &= -\psi^{p_2}\left(\sqrt{a_1}dt + \sqrt{-a_2}dx\right)^2 + \\ &\quad + \psi^{p_1}\left(\sqrt{-a_2}dt + \sqrt{a_1}dx^2\right)^2 + \\ &\quad e^{\beta}\left[dr^2 + r^2 d\sigma^2\right]\,. && (43)\end{aligned}$$

Now we see that the transformation:

$$T = \sqrt{a_1}t + \sqrt{-a_2}x\,, \quad (44)$$
$$X = \sqrt{-a_2}t + \sqrt{a_1}x \quad (45)$$

will give us the diagonal form of the metric:

$$-{}^{(5)}ds^2 = \psi^{p_2}dT^2 - \psi^{p_1}dX^2 - e^{\beta}\left[dr^2 + r^2 d\sigma^2\right], \quad (46)$$

So we have here an example of the removable electric field (charge).

In their work, authors claims that this is a general spherically symmetric solution of KK theory. However, in a general solution, there should be 4 independent parameters, more precisely: $m-$ mass, $q-$ electromagnetic charge, $g-$ scalar charge, $v-$ boost parameter. But here we have actually only three parameter: m, g, v, moreover, electromagnetic charge is actually some function of the rest of parameters, $q = q(v, m, g)$. So for the obtained electric field is an open question about the sense of the thus obtained electric field and charge

Frolov et al. (1987) carried out a formal embedding of the Kerr metric in the flat space M^5, and then applied a boost. By this procedure they obtaining electromagnetic field, and even the scalar field. Of course, those are trivial fields, that can be removed by another global boost. Note that those authors reference to works of Gibbons (1982), Gibbons and Wiltshire (1985), where this method been developed.

We note also the work Vladimirov and Popov (1982), which uses a similar method of generating an electric field.

Apart from the question of the interpretation of this removable electric field, there are still problems related to the relationship between the the closure condition space 5D and 5D boost. In addition to that, it is appropriate to ask if there are solutions where electromagnetic field A_0 is unremovable by this procedure, and with an independent charge parameter q.

3.2. Unremovable Electric Charge

In work Gladush (1980), there is a solution of spherically-symmetric configuration of interacting scalar, electromagnetic and gravitational fields of 5D KK theory, with the following reduced metric and Lagrangian:

$$\Lambda = -\frac{1}{4\pi}\left[\frac{c^4}{4\kappa}{}^{(4)}R - \frac{1}{2}\left(\nabla\psi\right)^2 + \right.$$
$$\left. + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}e^{\psi\sqrt{6}}\right], \quad (47)$$

$${}^{(5)}ds^2 = \frac{1}{2}e^{-\psi\sqrt{2/3}}\,{}^{(4)}ds^2 - e^{2\psi\sqrt{2/3}}\left(dz + f dt\right)^2, \quad (48)$$

where 4D metric is:

$${}^{(4)}ds^2 = e^{\nu}dt^2 - e^{-\nu}\left[dr^2 + \left(r^2 - a^2\right)d\sigma^2\right] \quad (49)$$

while the gravitational potential, scalar field, and electric field are:

$$e^{\nu} = u^{G\sqrt{3\kappa}/2a}\left[\frac{A+1}{2}u^{-p} - \frac{A-1}{2}u^{p}\right]^{-1/2} \quad (50)$$

$$\psi = -\frac{1}{4}\ln\left[u^{\frac{G}{a}}\left(\frac{A+1}{2u^{p}} - \frac{A-1}{2u^{-p}}\right)^{\sqrt{\frac{3}{\kappa}}}\right] \quad (51)$$

$$f = \frac{q}{2p}\frac{1-u^{2p}}{A+1-(A-1)\,u^{2p}}\,, \quad (52)$$

$$u = \frac{r-a}{r+a}\,, \quad p = \frac{\sqrt{\kappa^2 m^2 - \kappa q^2}}{a}\,, \quad (53)$$

$$A = \frac{\pm\kappa m}{\sqrt{\kappa^2 m^2 - \kappa q^2}}\,, \quad (54)$$

$$a^2 = \kappa^2 m^2 - \kappa q^2 + \kappa G^2\,, \quad \kappa^2 m^2 - \kappa q^2 > 0 \quad (55)$$

Here, we have three independent parameters $\{m,\ q,\ G\}$, and the forth parameter v can be included by applying some boost. There is no global transformation with constant coefficients, that can be used to remove the electromagnetic field, thus this field is unremovable.

In the work of Bronikov and Shikin (1977) a similar problem is solved for a system of interacting scalar, electromagnetic and gravitational fields, and with an action similar to above mentioned. But the field was interpreted not in the context of 5D theory. Also they used harmonic time gauge for the metric. However if we analyze it in the context of 5D theory, one finds that the electric field here as well is unremovable.

4. A symmetrical approach for constructing new solutions of 5D KK theory

For stationary spaces, in KK theory there is a method of constructing solutions by using internal symmetries of KK equations. This method has been developed in the papers of Maison Kramer and Neugebauer (1969), Maison (1979), Dobiasch and Maison (1982), Clement (1986), Cvetic and Youm (1995) and others.

The stationary metric(14) 5D KK theory in 2+3 splitting: be rewritten as:

$$ds^2 = \gamma_{ab}\left(x^k\right)dx^a dx^b - \frac{\tilde{h}_{ij}}{\tau}\left(x^k\right)dx^i dx^j\,, \quad (56)$$
$$\tau = \left|\det\|\gamma_{ab}\|\right|. \quad (57)$$

Then the 5D lagrangian for vacuum 5D space of KK theory is:

$$L = \sqrt{{}^{(5)}g}\,{}^{(5)}R\,,$$

by omitting the divergence, it can be rewritten as Lagrangian for the 4D space

$$L = \sqrt{\tilde{h}}\left\{ {}^{(3)}R - \frac{1}{4}\left[\gamma^{ab}\gamma^{cd}\gamma_{ac,k}\gamma_{bd,l}^{,k} + \gamma^2\left(\tau^{-1}\right)_{,k}\left(\tau^{-1}\right)_{,l}\right]\tilde{h}^{kl}\right\}. \quad (58)$$

Furthermore, we introduce a symmetrical unimodular matrix 3×3:

$$\chi = \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad (59)$$

such that $\det \chi = 1$. Then the Lagrangian can be rewritten as (σ-model):

$$L = \sqrt{\tilde{h}}\left\{ {}^{(3)}R - \frac{1}{4}Sp\left(\gamma^{-1}\gamma_{,k}\gamma^{-1}\gamma^{,k}\right)\right\}. \quad (60)$$

From this we get the following equations:

$$\left(\chi^{-1}\chi^{,k}\right)_{;k} = 0\,, \quad (61)$$

$${}^{(3)}R_{ij} = \frac{1}{4}Sp\left(\chi^{-1}\chi_{,i}\chi^{-1}\chi_{,j}\right). \quad (62)$$

This system is invariant under group $SL(3,R)$ transformations of 3D matrices:

$$\chi \longrightarrow N\chi N^T\,, \quad N \in SL(3,R)\,. \quad (63)$$

To force this metric γ_{ab} be asymptotically flat space, we should have at infinity:

$$\chi = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}. \quad (64)$$

Only the subgroup $SO(1,2)$ of the group $SL(3,R)$ can satisfy this property. Now we can generate new solutions by applying some transformation of the group SO(1,2) to a known solution. For the class of metrics under our consideration, we only need the transformations $\widetilde{SO}(1,2) \subset SO(1,2)$, which can conserve the block structure of the matrix χ.

We see that the analyzed Lorentz transformation $L(2) = O(1,1)$ is actually a subgroup of $\widetilde{SO}(1,2)$ $\tilde{x}^{\tilde{a}} = L_b^{\tilde{a}}x^b$. So we have:

$$O(1,1) \subset \widetilde{SO}(1,2) \subset SO(1,2) \subset SL(3,R)\,.$$

While the transformed matrix χ has the structure:

$$N = \begin{pmatrix} L_b^{\tilde{a}} & 0 \\ 0 & 1 \end{pmatrix}, \quad (65)$$

where $L_b^{\tilde{a}}$ are the 2D boosts matrix, that reduces to the Lorentz coordinate transformations. The set of solutions that can be obtained by this method is not very big. The group $O(1,1)$ that induced by coordinate transformation can give us solutions only with trivial electric field. Hence we see the place of coordinate transformations that generate the trivial electric fields, among the entire set of transformations SO(1,2), which in the general case generate a nontrivial field.

We also see that the physically significant transformations $\widetilde{SO}(1,2)$ – are a map of one KK space (solution) to another one, that can not be reduced to simple coordinate transformations.

5. Conclusion

We observed that a phenomenon of generating an electric field can be interpreted as 5D rotation. In general, electric field is a result of local 5D rotations of 5D manifold (i.e. existence of local inertial 5D reference frames), that defined by additional local dynamical degrees of freedom. The corresponding integral of motion gives the conserved charge, as an independent parameter, This is the case of non trivial unremovable electrical field (charge).

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