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SINGULARITIES AND THEIR CROSSING IN GRAVITY AND COSMOLOGY

A. Yu. Kamenshchik

Department of Physics and Astronomy "Augusto Righi", University of Bologna, National Istitute for Nuclear Physics, Bologna, Italy, *kamenshchik@bo.infn.it*

ABSTRACT. We discuss the problem of singularity crossing in isotropic and anisotropic universes. First, we consider the so called soft or sudden singularities and, in particular the Big Brake singularity. This singularity was discovered in a particular tachyon cosmological model and it was also shown that this kind of singularity arises in a very simple model, where matter is represented by the anti-Chaplygin gas. At the encounter with the Big Brake singularity the universe has a finite scale factor, a vanishing expansion velocity and an infinite deceleration. The Christoffel symbols also vanish the geodesics are regular and the universe easily can cross such a singularity. Adding to the anti-Chaplygin gas or to the tachyon matter some amount of dust we see that the Big Brake singularity is substituted by a more general soft singularity, its crossing implies a certain transformation of the properties of matter. The crossing of the Big Bang – Big Crunch singularity is more counter-intuitive. However, we describe it for both Friedmann universe and Bianchi-I universe using the field reparametrization of the variables present in models (a scalar field and the metric). Then we consider the Wheeler-DeWitt equation and show that the probability for the universe to find itself at the soft singularity is different from zero, while the encounter with the Big Bang – Big Crunch singularity is suppressed. We analyze the possibility to construct Fock spaces of quantum particles at the vicinity of different cosmological singularities and see when it is possible and when it is not possible. Finally, we present some attempts to develop general approach to the connection between the field reparametrization and the elimination of singularities.

Keywords: gravitation; cosmology; singularities.

АНОТАЦІЯ. Ми обговорюємо проблему перетину сингулярності в ізотропних та анізотропних всесвітах. Спочатку ми розглянемо так звані м'які або раптові сингулярності та, зокрема, сингулярність Великого Гальма. Ця сингулярність була виявлена в певній тахіонній космологічній моделі, і було також показано, що цей тип сингулярності виникає в дуже простій моделі, де матерія представлена античаплигінським газом. При зіткненні з сингулярністю Великого Гальма Всесвіт має скінченний масштабний коефіцієнт, зникаючу швидкість розширення та нескінченне уповільнення. Символи Крістоффеля також зникають, геодезичні є регулярними, і Всесвіт може легко перетнути таку сингулярність. Додаючи до античаплигінського газу або тахіонної матерії певну кількість пилу, ми бачимо, що сингулярність Великого Гальма замінюється більш загальною м'якою сингулярністю, її перетин передбачає певну трансформацію властивостей матерії. Перетин сингулярності Великого Вибуху – Великого Стиснення є більш контрінтуїтивним. Однак, ми описуємо це як для Всесвіту Фрідмана, так і для Всесвіту Б'янкі-I, використовуючи репараметризацію поля змінних, присутніх у моделях (скалярне поле та метрика). Потім ми розглядаємо рівняння Уілера-Девітта і показуємо, що ймовірність того, що Всесвіт опиниться в м'якій сингулярності, відрізняється від нуля, тоді як зустріч із сингулярністю Великого вибуху – Великого стиснення виключається. Ми аналізуємо можливість побудови просторів Фока квантових частинок поблизу різних космологічних сингулярностей та бачимо, коли це можливо, а коли ні. Нарешті, ми представляємо деякі спроби розробити загальний підхід до зв'язку між репараметризацією поля та усуненням сингулярностей.

Ключові слова: гравітація; космологія; сингулярність.

1. Introduction

Appearance of singularities is one of the most important phenomena in General Relativity and in its generalizations and modifications. The singularities were first discovered in such simple geometries as those of Friedmann and Schwarzschild and later their general character was established in (Penrose, 1965; Hawking, 1966; Gorini et al., 2004). The investigation of the oscillatory approach to the cosmological singularity (Belinsky, Khalatnikov & Lifshitz, 1970) known also as Mixmaster universe (Misner, 1969) has opened the way to the birth of a new branches of the mathematical physics – chaotic cosmology (Khalatnikov, Lifshitz, Khanin et al., 1985) and its relation tohyperbolic Kac-Moody algebras (Damour, Henneaux & Nicolai, 2003).

While some researchers try to exclude cosmological singularities and singularities hidden inside black holes, constructing some involved models, the idea that the singularities are not a drawback of the General Relativity but is natural and fundamental feature becomes more popular during new millennium. Remarkably this idea was advocated by Misner (Misner, 1969a) as early as in 1969. Let us give some direct citations from his enchanting paper.

"I prefer a more optimistic viewpoint ("Nature and Einstein are subtle but tolerant") which views the initial singularity in cosmological theory not as a proof of our ignorance, but as a source from which we can derive much valuable understanding of cosmology."

"Thus, while I presume that relativity, like other physical theories, will be improved from time to time, I do not see that these changes need bear directly on the problem of cosmological singularity."

"We should stretch our minds, find some more acceptable set of words to describe the mathematical situation, now identified as "singular", and then proceed to incorporate this singularity into our physical thinking until observational difficulties force revision on us."

"The concept of a true initial singularity (as distinct from an indescribable early era at extravagant but finite high densities and temperatures) can be a positive and useful element in cosmological theory.
The Universe is meaningfully infinitely old because infinitely many things have happened since the beginning."

Inspired by this spirit of treatment of the singularities as something natural, one can try to study the opportunity to cross them. In this paper, based on my talk at the XXV Gamov International Astronomical Conference, I shall review different aspects of the singularity crossing in gravity and cosmology. The structure of the paper is the following: the second section is devoted to the description of the so called soft or sudden singularities; in the third section we treat the more traditional Big Bang – Big Crunch singularity; the fourth section is devoted to quantum cosmology; in the fifth section we discuss what happens with quantum particles at the singularity crossing; in the sixth section we describe attempts to develop a general approach to the description of the singularity crossing; the last section contains conclusive remarks.

2. The Big Brake cosmological singularity, more general soft singularities and their crossing

We start this section with the consideration of a very particular toy model (Gorini, Kamenshchik, Moschella et al., 2004). A flat Friedmann universe with the metric

$$ds^2 = dt^2 - a^2(t)dl^2$$

is driven by the so called tachyon field (Sen 2002) with the Lagrange density

$$L = -V(T)\sqrt{1-\dot{T}^2}.$$

The energy density is

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}},$$

the pressure is

$$p = -V(T)\sqrt{1-\dot{T}^2}.$$

The Friedmann equation has the following form:

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \rho.$$

The equation of motion for the tachyon field is

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V} = 0.$$

In our model the potential is

$$V(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]} \times\sqrt{1-(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\,T\right]}\;,$$

where k and $\Lambda > 0$ are the parameters of the model. The case $k > 0$ is more interesting. Indeed, in this case some trajectories (cosmological evolutions) finish in an infinite de Sitter expansion. In other trajectories the tachyon field transforms into a pseudotachyon field with the Lagrange density, energy density and positive pressure (Gorini et al., 2004):

$$\begin{aligned} L &= W(T)\sqrt{\dot{T}^2-1}, \\ \rho &= \frac{W(T)}{\sqrt{\dot{T}^2-1}}, \\ p &= W(T)\sqrt{\dot{T}^2-1}, \\ W(T) &= \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\;T\right]} \\ &\times\sqrt{(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+k)}\,T-1\right]} \end{aligned}$$

What happens to the Universe after the transformation of the tachyon into the pseudotachyon ? It encounters the Big Brake cosmological singularity.

The Big Brake cosmological singularity has the following characteristics:

$$t \to t_{BB} < \infty,$$

$$a(t \to t_{BB}) \to a_{BB} < \infty,$$

$$\dot{a}(t \to t_{BB}) \to 0,$$

$$\ddot{a}(t \to t_{BB}) \to -\infty,$$

$$R(t \to t_{BB}) \to +\infty,$$

$$\rho(t \to t_{BB}) \to 0,$$

$$p(t \to t_{BB}) \to +\infty.$$

If $\dot{a}(t_{BB}) \neq 0$ it is a more general soft singularity.

At the Big Brake singularity the equations for geodesics are regular, because the Christoffel symbols are regular (moreover, they are equal to zero). We can ask ourselves if it is possible to cross the Big Brake singularity (Gorini et al., 2004). Let us study the regime of approaching to the Big Brake. On analyzing the equations of motion we find that on approaching the Big Brake singularity the tachyon field behaves as

$$T = T_{BB} + \left(\frac{4}{3W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{1/3}.$$

Its time derivative $s \equiv \dot{T}$ behaves as

$$s = -\left(\frac{4}{81W(T_{BB})}\right)^{1/3} (t_{BB} - t)^{-2/3},$$

the cosmological radius is

$$a = a_{BB} - \frac{3}{4}a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{4/3},$$

its time derivative is

$$\dot{a} = a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{1/3},$$

and the Hubble variable is

$$H = \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (t_{BB} - t)^{1/3}.$$

All these expressions can be continued into the region where $t > t_{BB}$, which amounts to crossing the Big Brake singularity. Only the expression for s is singular at $t = t_{BB}$ but this singularity is integrable and not dangerous.

Once reaching the Big Brake, it is impossible for the system to stay there because of the infinite deceleration, which eventually leads to a decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius and Hubble variable change their signs. The expansion is then followed by a contraction, culminating in the Big Crunch singularity.

One of the simplest cosmological models revealing a Big Brake singularity is the model based on the anti-Chaplygin gas with an equation of state

$$p = \frac{A}{\rho}, \quad A > 0.$$

Such an equation of state arises in the theory of wiggly strings (Carter, 1989; Vilenkin, 1990). Here

$$\rho(a) = \sqrt{\frac{B}{a^6} - A}.$$

At $a = a_* = \left(\frac{B}{A}\right)^{1/6}$ the universe encounters the Big Brake singularity.

What happens in a universe filled with an anti-Chaplygin gas and dust (Keresztes et al., 2013; Gorini et al., 2004)? The energy density and the pressure are

$$\rho(a) = \sqrt{\frac{B}{a^6} - A} + \frac{M}{a^3}, \quad p(a) = \frac{A}{\sqrt{\frac{B}{a^6} - A}}.$$

Due to the dust component, the Hubble parameter has a non-zero value at the encounter with the singularity, therefore the dust implies further expansion. However, with continued expansion however, the energy density and the pressure of the anti-Chaplygin gas would become ill-defined. The abrupt transition from the expansion to the contraction of the universe does not look natural. One can try to change the equation of state of the anti-Chaplygin gas on passing the soft singularity. There is some analogy between the transition from an expansion to a contraction of a universe and the perfectly elastic bounce of a ball from a wall in classical mechanics. There is also an abrupt change of the direction of the velocity (momentum). However, we know that in reality the velocity is changed continuously due to the deformation of the ball and of the wall. The pressure of the anti-Chaplygin gas

$$p = \frac{A}{\sqrt{\frac{B}{a^6} - A}}$$

tends to $+\infty$ when the universe approaches the soft singularity. Requiring the expansion to continue into the region $a > a_S$, while changing minimally the equation of state, we assume

$$p = \frac{A}{\sqrt{|\frac{B}{a^6} - A|}},$$

$$p = \frac{A}{\sqrt{A - \frac{B}{a^6}}}, \text{ for } a > a_S.$$

This implies the energy density

$$\rho = -\sqrt{A - \frac{B}{a^6}}.$$

The anti-Chaplygin gas transforms itself into Chaplygin gas with negative energy density. The pressure remains positive, expansion continues. The spacetime geometry remains continuous. The expansion stops at $a = a_0$, where

$$\frac{M}{a_0^3} - \sqrt{A - \frac{B}{a_0^6}} = 0.$$

Then the contraction of the universe begins. At the moment when the energy density of the Chaplygin gas becomes equal to zero (again a soft singularity), the Chaplygin gas transforms itself into the anti-Chaplygin gas and the contraction continues culminating in an encounter with the Big Crunch singularity. Analogous effects arise in the model with the tachyon field and dust. The Lagrangian of the Born-Infeld like field changes its form.

We conclude this section by mentioning that the soft (sudden) cosmological singularities were firstly studied in (Barrow, Galloway & Tipler 1986). The conditions of the singularity crossing were studied in (Fernandez-Jambrina & Lazkoz 2004). Interesting tachyon cosmological models were suggested in (Feinstein 2002) and (Padmanabhan 2002).

3. Big Bang – Big Crunch singularity crossing

The idea that the Big Bang – Big Crunch singularity can be crossed appears very counterintuitive (Gorini et al., 2004). Some approaches to the description of this crossing were elaborated during the last two decades (Bars et al., 2012; Wetterich, 2014; Dominis Prester, 2016). There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization. On choosing some convenient field parametrization one can provide a matching between the characteristics of the universe before and after the singularity crossing. One can trace an analogy to the Kruskal coordinates for the Schwarzschild metric. On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang – Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.

Let us consider a flat Friedmann universe filled with a conformally coupled scalar field (Kamenshchik, Pozdeeva, Tronconi et al., 2016).

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + V(\sigma)\right],$$

$$U(\sigma) = U_0 - \frac{1}{12}\sigma^2.$$

Let us apply the conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_1}{U}\tilde{g}_{\mu\nu}.$$

A new scalar field ϕ is

$$\frac{d\phi}{d\sigma} = \frac{\sqrt{U_1(U + 3U'^2)}}{U} \Rightarrow \phi = \int \frac{\sqrt{U_1(U + 3U'^2)}}{U}d\sigma.$$

$$\phi = \sqrt{3U_1}\ln\left[\frac{\sqrt{12U_0} + \sigma}{\sqrt{12U_0} - \sigma}\right]$$

$$\sigma = \sqrt{12U_0}\tanh\left[\frac{\phi}{\sqrt{12U_1}}\right].$$

The action then becomes the action for a minimally coupled scalar field:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[U_1R(\tilde{g}) - \frac{1}{2}\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + W(\phi)\right],$$

$$W(\phi) = \frac{U_1^2V(\sigma(\phi))}{U^2(\sigma(\phi))}.$$

This is called the transformation from the Jordan frame to the Einstein frame.

In a flat Friedmann universe

$$ds^2 = N^2d\tau^2 - a^2dl^2,$$

$$d\tilde{s}^2 = \tilde{N}^2d\tau^2 - \tilde{a}^2dl^2.$$

$$\tilde{N} = \sqrt{\frac{U}{U_1}}N,\ \tilde{a} = \sqrt{\frac{U}{U_1}}a,\ t = \int\sqrt{\frac{U_1}{U}}d\tilde{t},$$

where t and $\tilde{t}$ are the cosmic time parameters in the Jordan and the Einstein frames.

$$a = \tilde{a}\sqrt{\frac{U_1}{U_0}}\cosh\left(\frac{\phi}{\sqrt{12U_1}}\right).$$

In the vicinity of the singularity in the Einstein frame:

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}} \to 0, \text{ when } \tilde{t} \to 0.$$

However, in the Jordan frame one has

$$a \sim \tilde{t}^{\frac{1}{3}}\left(\tilde{t}^{\frac{1}{3}} + \tilde{t}^{-\frac{1}{3}}\right) \to \text{const} \neq 0.$$

Meanwhile, the scalar field σ crosses the value $\pm\sqrt{12U_0}$ and the coupling function U changes its sign. Thus, the evolution in the Jordan frame is regular, and we can use this fact to describe the crossing of the Big Bang – Big Crunch singularity in the Einstein frame. If one considers the expansion of the universe from the Big Bang with normal gravity driven by the standard scalar field, the continuation backward in time shows that it was preceded by the contraction towards a Big Crunch singularity in the antigravity regime, driven by a phantom scalar field with a negative kinetic term.

The possibility of a change of sign of the effective gravitational constant in the model with a conformably coupled scalar field was analyzed in (Starobinsky, 1981; Gorini et al., 2004). It was shown that in a homogeneous and isotropic universe, one can indeed cross the point where the effective gravitational constant changes sign. However, the presence of anisotropies changes the situation: these anisotropies grow indefinitely when this constant is equal to zero. To describe the Big Bang – Big Crunch singularity crossing in anisotropic universes it is necessary to use another methods. We have done it for the Bianchi-I universe (Kamenshchik, Pozdeeva, Tronconi et al 2017, Kamenshchik, Pozdeeva, Starobinsky et al., 2018). The idea was the following. Introducing the new "radial" variable

$$r \sim a^{3/2}$$

and treating the scalar field as an angular variable, we obtain an effective Lagrangian

$$L = \frac{1}{2}\dot{r}^2 - \frac{1}{2}r^2\dot{\phi}^2,$$

or, in Cartesian coordinates

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2.$$

Solving the equations of motion one can describe the singularity crossing. Introduction of the anisotropy factors for the Bianchi-I model is reduced to the modification of the kinetic term for the massless scalar field.

4. Quantum cosmology and singularities

Speaking about quantum cosmology and singularities people mean two different approaches. One can consider a modification of the Friedmann equation, taking into account the quantum corrections to the effective action of the theory:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \rho_{\text{matter}} + \rho_{\text{quantum corrections}}.$$

One can hope that these correction change the dynamics of the universe evolution, implying an appearance of some kind of bounce and avoing an encounter with the singularity. Another approach is based on the study of the Wheeler-DeWitt equation and the prospective of vanishing of the quantum state of the universe at the singular configurations of the geometry (DeWitt 1967):

$$\Psi(\text{geometry} + \text{matter})_{\text{geometry is singular}} = 0.$$

The wave function of the Universe Ψ satisfies the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0,$$

where $\hat{\mathcal{H}}$ is the so called super-Hamiltonian. There are two major questions, concerning this equation. It looks like the very notion of time disappears here. Then, it is not clear how the notion of the probability can be determined. There is rather a vast literature devoted to treatment of these problems (see e.g. (Barvinsky 1993)). The general recipe for their treatment can be formulated as follows: a time can be defined as a certain function of geometrical variables. After that the wavefunction describing matter variables satisfies an effective Schrödinger equation. The singularity is associated with such values of the matter variables when this singularity arises in the classical theory. Our analysis of some simple models tells that the probability of the arising of soft singularities is not suppressed by the wave function of the universe (Kamenshchik, Kiefer & Sandhofer 2007, Kamenshchik and Manti 2012, Kamenshchik, Kiefer & Kwidzinski 2016). At the same time the probability of Big Bang – Big Crunch singularity tends to zero. The suppression of the Big Bang – Big Crunch singularity follows from the requirement of the normalizability of the wave function of the Universe (Barvinsky & Kamenshchik 1990). Indeed, we require that

$$\int d\phi \bar{\Psi}(\phi)\Psi(\phi) < \infty,$$

where ϕ is a scalar field, driving the evolution of the universe. When $|\phi| \to \infty$, the probability density $\bar{\Psi}\Psi$ should tend to zero rapidly. If $|\phi| \to \infty$ corresponds to Big Bang – Big Crunch singularity, when this singularity is suppressed.

5. Particles, fields and singularities

We can ask ourselves what happens with particles (in quantum field theoretical sense) when the universe passes through the cosmological singularity (Galkina & Kamenshchik 2020). We consider only particles connected with a scalar field. The scalar field in the flat Friedmann universe satisfies the Klein-Gordon equation:

$$\Delta\phi + V'(\phi) = 0,$$

where Δ is the D'Alambert operator. One can consider a spatially homogeneous solution of this equation ϕ_0, depending only on time t as a classical background. A small deviation from this background solution can be represented as a sum of Fourier harmonics satisfying linearized equations

$$\ddot{\phi}(\vec{k},t) + 3\frac{\dot{a}}{a}\dot{\phi}(\vec{k},t) + \frac{\vec{k}^2}{a^2}\phi(\vec{k},(t)) + V''(\phi_0(t))\phi(\vec{k},(t)) = 0.$$

The corresponding quantized field is

$$\hat{\phi}(\vec{x},t) = \int d^3\vec{k}(\hat{a}(\vec{k})u(k,t)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+(\vec{k})u^*(k,t)e^{-i\vec{k}\cdot\vec{x}}),$$

where the creation and the annihilation operators satisfy the standard commutation relations:

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \delta(\vec{k} - \vec{k}').$$

The basis functions should be normalized so that the canonical commutation relations between the field ϕ and its canonically conjugate momentum $\hat{\mathcal{P}}$ were satisfied

$$[\hat{\phi}(\vec{x},t), \hat{\mathcal{P}}(\vec{y},t')] = i\delta(\vec{x} - \vec{y}).$$

$$u(k,t)\dot{u}^*(k,t) - u^*(k,t)\dot{u}(k,t) = \frac{i}{(2\pi)^3 a^3(t)}.$$

The linearized Klein-Gordon equation has two independent solutions. To define a particle it is necessary to have two independent non-singular solutions. It is a non-trivial requirement in the situations when a singularity or other kind of irregularity of the spacetime geometry occurs. It is convenient also to construct explicitly the vacuum state for quantum particles as a Gaussian function of the corresponding variable. Let us introduce an operator

$$\hat{f}(\vec{k},t) = (2\pi)^3(\hat{a}(\vec{k})u(k,t) + \hat{a}^+(-\vec{k})u^*(k,t)).$$

Its canonically conjugate momentum is

$$\hat{p}(\vec{k},t) = a^3(t)(2\pi)^3(\hat{a}(\vec{k})\dot{u}(k,t) + \hat{a}^+(-\vec{k})\dot{u}^*(k,t)).$$

We can express the annihilation operator as

$$\hat{a}(\vec{k}) = i\hat{p}(\vec{k},t)u^*(k,t) - ia^3(t)\hat{f}(\vec{k},t)\dot{u}^+(k,t).$$

Representing the operators $\hat{f}$ and $\hat{p}$ as

$$\hat{f} \to f, \quad \hat{p} \to -i\frac{d}{df},$$

one can write down the equation for the corresponding vacuum state in the following form:

$$\left(u^*\frac{d}{df} - ia^3\dot{u}^* f\right)\Psi_0(f) = 0.$$

Its solution is

$$\Psi_0(f) = \frac{1}{\sqrt{|u(k,t)|}}\exp\left(\frac{ia^3(t)\dot{u}^*(k,t)f^2}{2u^*(k,t)}\right).$$

In the case of the Big Bang – Big Crunch singularity, one of the basis functions in the vicinity of the singularity becomes singular and it is impossible to construct a Fock space. In the case of the Big Rip singularity, when in finite interval of time the universe achieves an infinite volume and infinite time derivative of the scale factor, the Fock space can be constructed for a spectator scalar field, but it does not exist for the phantom scalar field driving the expansion. In the case of the model with tachyon field, presented above, we have considered three situations. First of them is the non-singular transformation of the tachyon into pseudo-tachyon. In this case both the basis functions are regular and hence the operators of creation and annihilation are well defined. However, at the moment of the transformation the dispersion of the Gaussian wave function of the vacuum becomes infinite and then becomes finite again. In the vicinity of the Big Brake singularity it is impossible to define a Fock vacuum. However, if we add to the universe dust, the character of the soft singularity is slightly changed and then the presence of the Fock vacuum is restored.

6. Covariant approach to singularities

We have already told that the crossing of the Big Bang – Big Crunch singularities looks rather counterintuitive. However, it can be sometimes described by using the reparametrization of fields, including the metric. One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof. One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular. Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge. The application of these ideas looks in a way as a craftsman work. Our goal was to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration. In other words, we try to understand when the spacetime singularities can be removed by a reparametrization of the field variables (Casadio, Kamenshchik & Kuntz, 2021; Casadio, Kamenshchik & Kuntz, 2022; Kuntz, Casadio & Kamenshchik, 2022).

Our hypothesis was the following: when the geometry of the space of the field variables is non-singular, one can describe the singularity crossing. The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory (Vilkovisky, 1984; DeWitt, 1987).

This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G. G is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter. After some cumbersome calculations in the functional space, we have shown that the Kretschmann scalar

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}$$

is finite in every theory of pure gravity

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right),$$

where n is the spacetime dimension. It can be interpreted as a statement that all the singularities in empty universe can be crossed.

We have considered also another hypothesis connected to quantum effective action and to the homotopy group. Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]},$$

where $\Gamma[\varphi]$ is the effective action. We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role to the order parameter in the theory of phase transitions in ordered media or cosmology.

The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects.

The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to S^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space. The singularities can be characterized by the fundamental group (first homotopy group). If this group is trivial the singularity can be removed. We have checked on the example of some simple systems with removable singularity that the corresponding homotopy group is indeed trivial.

7. Conclusions

In this paper we have tried to present some arguments and results telling that the appearance of the singularities in the cosmological and other gravitational systems is not drawback of models or theories but is instead their distinguishing feature. Thus, to our mind rather than avoid singularities, it is better to study how their presence influences the non-singular quantities (just like in quantum field theory). Further details and references can be found in the review papers (Gorini et al., 2004; Kamenshchik, 2013; Kamenshchik, 2018; Kamenshchik, 2024).

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