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SPHERICALLY SYMMETRIC SYSTEM OF GRAVITATIONAL AND ELECTROMAGNETIC FIELDS AND THE STRUCTURE OF ITS CONFIGURATION SPACE

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ABSTRACT. Geometrodynamics of charged black holes (BH) described by the system of Maxwell-Einstein equations is considered. We start from a spherically symmetric metric, a reduced action, and a Lagrangian written in characteristic variables. The configuration space (CS) metric, Hamiltonian, momentum and electromagnetic constraints are constructed. The system has conservation laws of charge q and mass m. The action functional is transformed into a Jacobi-type functional in CS with a metric conformal to the CS metric. A transformation of field variables is introduced which brings the CS metric to the "Lorentzian"form. The resulting CS metric is the metric of a flat nonholonomic section of a 4-dimensional space. In the new variables, the squared momenta of the system has the Lorentz form. On this basis, quantization is considered. Thanks to the structure of the CS, the momentum operators, the DeWitt equations, and the mass and charge operators are constructed. The equations system of CBH quantum states with certain q and m is constructed. For comparison, we consider the CBH reduced model limited in the T-region. In such the simplified formulation, the T-model equations are integrated and lead to the CBH with continuous spectrum of m and q.

Keywords: spherically symmetric configurations, configuration space, Hamiltonian constraint, DeWitt operators, mass and charge, quantization, charged black holes.

АНОТАЦІЯ. Розглядається загальний підхід до геометродинаміки заряджених чорних дірок (ЗЧД), що описуються сферично-симетричними конфігураціями гравітаційного та електромагнітного полів. Ми виходимо з метрики, редукованої дії та лагранжіана, записаних у характеристичних змінних. Вводяться узагальнені швидкості та метрика конфігураційного простору (КП). Будуються гамільтонова, імпульсна та електромагнітна в'язі. Система має закони збереження заряду q та маси m. Використовуючи гамільтонову в'язь та закони збереження, знаходяться вирази для імпульсів через конфігураційні змінні та q і m. З рівнянь для імпульсів у функціональних похідних від дії будується функціонал дії. Похідні дії по q і m призводять до рівнянь траєкторій в КП. Далі функціонал дії перетворюється на функціонал дії типу Якобі в КП з метрикою, конформної до метрики КП. Вводиться перетворення польових змінних, які зводять метрику КП до "лоренцевого"виду. Це приводе нелінійну систему рівнянь ЗЧД до лінійної, де всі компоненти поля поділяються. Одержана метрика КП може розглядатися як метрика плоского неголономного перерізу 4-вимірного простору. В нових змінних квадрат імпульсів системи має теж Лоренців вигляд. На цій основі розглядається квантування системи. Завдяки структурі КП вдається побудувати коректні оператори імпульсів, рівняння Девітта та оператори маси та заряду. Будується система рівнянь у функціональних похідних для квантових станів ЗЧД із певними q і m. Для порівняння розглядається редукована модель ЗЧД, обмеженої в Т-області. У такій спрощеній постановці рівняння Т-моделі інтегруються і призводять до моделі ЗЧД із безперервними спектрами m і q. Побудова редукована Т-модель намічає шляхи подальшого дослідження загальної системи рівнянь квантової геометродинаміки ЗЧД.

Ключові слова: сферично-симетричні конфігурації, конфігураційний простір, Гамільтонова в'язь, оператори Девітта, маси і заряду, квантування, заряджені чорні діри.

1. Introduction

The paper is devoted to the study of the classical and quantum aspects of the geometrodynamics of charged black holes (CBH), the research on the structure of their configuration space (CS) and the construction of a system of quantum equations in functional derivatives that describe the model of CBH.

Geometrodynamics of CBH is described by the Einstein equations system for a spherically symmetric configuration of the gravitational and electromagnetic fields in GR. As is known, the space-time metric $g_{\mu\nu}$ of such a configuration of fields admits the Killing vector. The region $R \subset \mathbf{M}^{(4)}$, where this vector is timelike, is called the R-region, while the region $T \subset \mathbf{M}^{(4)}$, where this vector is spacelike, is called the T-region [Gladush 21].

First, consider the configuration of fields in the whole $\mathbf{M}^{(4)} = T \bigcup R$. We proceed from the following standard general action for the system of gravitational and electromagnetic fields in GR [Louko 1996, Mäkelä 1998]

$$S_{tot} = -\frac{1}{16\pi c} \int \left(\frac{c^4}{\kappa} \ {}^{(4)}R + F^{\mu\nu} F_{\mu\nu} \right) \sqrt{-g} d^4x \,, \quad (1)$$

where ${}^{(4)}R$ is the scalar curvature, κ is the gravitational constant, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor, $d^4x = dx^0 dx^1 dx^2 dx^3$, $g = \det |g_{\mu\nu}|$.

2. Classical geometrodynamics of charged BH

For non-rotating spherically symmetric configurations, we consider the space-time metric $\mathbf{M}^{(4)}$ and the electromagnetic field of the type [Gladush 19]

$$ds^2 = \frac{R}{\xi} \left(N dx^0 \right)^2 - \frac{\xi}{R} \left(dr + N^r dx^0 \right)^2 - R^2 d\sigma^2, \quad (2)$$

$$E = F_{01} = A_{r,0} - A_{0,r} = \phi_{,0} - \varphi_{,r}, \quad (3)$$

where $d\sigma^2 = d\theta^2 + \sin^2 \theta d\alpha^2$. Field configuration variables

$$q^A = \left\{ q^1 = q^R = R, \ q^2 = q^\xi = \xi, \ q^3 = q^\phi = \phi \right\} \quad (4)$$

are generalized coordinates that depend on space-time coordinates x^0, r, besides $A, B = \{1, 2, 3\}$.

The action S_{tot} after dimensional reduction, can be written as [Gladush 19]:

$$S_{tot} = \int dx^0 \int dr \mathcal{L} \,, \qquad \mathcal{L} = \frac{V^2}{2N} + N\mathcal{U} \quad (5)$$

where

$$V^2 = \Gamma_{AB} V^A V^B = -\frac{c^3}{\kappa} V^R V^\xi + \frac{1}{c} R^2 \left(V^\phi \right)^2 \quad (6)$$

is the velocity square of the kinetic part of the Lagrangian. Here Γ_{AB} are the covariant components of the CS metric, and $\Gamma = \det \|\Gamma_{AB}\| = -(c^5/4\kappa^2) R^2$, $V^A = q^A_{,0} + K^A$ – are the generalized velocity components

$$V^R = R_{,0} + K^R, \ V^\xi = \xi_{,0} + K^\xi, \ V^\phi = \phi_{,0} + K^\phi \quad (7)$$

where

$$K^R = -N^r R_{,r}, K^\xi = -\xi_{,r} N^r - 2\xi N^r_{,r}, K^\phi = -\varphi_{,r} \quad (8)$$

$\mathcal{U}$ is the potential part of the Lagrangian

$$\mathcal{U} = \frac{c^3}{2\kappa} \left(1 + \frac{R^2}{\xi^2} R_{,r} \xi_{,r} - \frac{2R}{\xi} R^2_{,r} - \frac{2R^2}{\xi} R_{,rr} \right) \quad (9)$$

Let us introduce the CS metric of the by the formula

$$d\Omega^2 = \Gamma_{AB} V^A V^B \left(dx^0 \right)^2 = \Gamma_{AB} Dq^A Dq^B \quad (10)$$

Here $Dq^a = dq^A + K^A dx^0$ are the Lie differentials

$$DR = dR + K^R dx^0, D\xi = d\xi + K^\xi dx^0, D\phi = d\phi + K^\phi dx^0$$

At that, the Γ_{AB} components are defined in (6). Then

$$d\Omega^2 = -\frac{c^3}{\kappa} D\xi DR + \frac{1}{c} R^2 \, D\phi^2 \quad (11)$$

The Legendre transformation of the system leads to the Hamiltonian action [Louko 1996]

$$\begin{aligned} S \ = \ & \int dx^0 \int_0^\infty dr \ \{ \mathcal{P}_\xi \xi_0 + \mathcal{P}_R R_{,0} + \mathcal{P}_\phi \phi_{,0} \\ & -N\mathcal{H} - N^r \mathcal{H}_r - \varphi \mathcal{H}_\phi \} \end{aligned} \quad (12)$$

where

$$\mathcal{H} = -\frac{2\kappa}{c^3} \mathcal{P}_R \mathcal{P}_\xi + \frac{c}{2R^2} \mathcal{P}^2_\phi - \mathcal{U} \sim 0 \,, \quad (13)$$

$$\mathcal{H}_r = -\xi_{,r} \mathcal{P}_\xi - 2\xi \mathcal{P}_{\xi,r} + R_{,r} \mathcal{P}_R \sim 0 \,, \quad (14)$$

$$\mathcal{H}_\phi = -\mathcal{P}_{\phi,r} \sim 0 \,. \quad (15)$$

so that $\mathcal{H}$ is Hamiltonian, $\mathcal{H}_r$ is momentum, and $\mathcal{H}_\phi$ is electromagnetic constraints expressed in terms of momenta. For convenience, we represent the Hamiltonian constraint in the form

$$H = \frac{1}{2} \mathcal{P}^2 - \mathcal{U} \quad (16)$$

where

$$\mathcal{P}^2 = \Gamma^{AB} \mathcal{P}_A \mathcal{P}_B = -\frac{4\kappa}{c^3} \mathcal{P}_R \mathcal{P}_\xi + \frac{c}{R^2} \mathcal{P}^2_\phi \quad (17)$$

is the momentum square. Note that Γ^{AB} are the contravariant components of the metric in the CS introduced earlier in (6) so that $\Gamma_{AB} \Gamma^{AD} = \delta^D_B$.

Electromagnetic constraint (15) determines the electric field E generated by the charge Q according to the formula

$$\mathcal{P}_\phi = \frac{R^2}{cN} E = \frac{Q}{c} = const \Rightarrow E = N \frac{Q}{R^2} \quad (18)$$

The system admits the motion integrals: the total mass M_{tot} and the charge $Q = c\mathcal{P}_\phi$ of configuration. The mass is determined by the mass function, which in terms of momenta has the form (Gladush 2012, 2018)

$$M_{tot} = \frac{c^2}{2\kappa}\left(R + \frac{4\kappa^2}{c^6}\xi\mathcal{P}_\xi^2 - \frac{R^2}{\xi}R_r^2\right) + \frac{\mathcal{P}_\phi^2}{2R} \tag{19}$$

Using the Hamiltonian constraint and conservation laws, one can find analytical expressions for momenta as functions of configuration variables and parameters m and q. Indeed, using the relations (18), (19) and (13), we obtain

$$\mathcal{P}_\xi = \frac{c^3}{2\kappa}\sqrt{\frac{R}{\xi}F_{tot}}\,, \tag{20}$$

$$\mathcal{P}_R = \sqrt{\frac{\xi}{RF_{tot}}}\left(\frac{q^2}{2cR^2} - \mathcal{U}\right). \tag{21}$$

where

$$F_{tot} = \frac{R}{\xi}\left(R_r\right)^2 - 1 + \frac{2\kappa m}{Rc^2} - \frac{\kappa q^2}{c^4R^2} \tag{22}$$

The momenta obtained in this way identically satisfy the invariance condition of the action functional, i.e. momentum constraint (14).

Using implicitly the integrability conditions of functional equations

$$\mathcal{P}_R = \frac{\delta S}{\delta R},\quad \mathcal{P}_\xi = \frac{\delta S}{\delta \xi},\quad \mathcal{P}_\phi = \frac{\delta S}{\delta \phi} = \frac{q}{c}\,, \tag{23}$$

we find the action functional S as a solution of the Einstein-Hamilton-Jacobi equation in functional derivatives depending on the variables R, ξ and parameters$m$ and q [Gladush 19]:

$$S = \int dr\left\{g\left(m,q;r\right) + \frac{Q}{c}\phi + \frac{c^3}{\kappa}\left(\sqrt{\xi RF_{tot}} - \frac{1}{2}RR_r\ln\left|\frac{RR_r + \sqrt{\xi RF_{tot}}}{RR_r - \sqrt{\xi RF_{tot}}}\right|\right)\right\} \tag{24}$$

Variations of S with respect to mass m and charge q lead to motion trajectories in the CS

$$\frac{\delta S}{\delta m} = -c\frac{\sqrt{R\xi F_{tot}}}{FR} + \frac{\partial g}{\partial m} = 0, \tag{25}$$

$$\frac{\delta S}{\delta q} = \frac{\sqrt{R\xi F_{tot}}}{cFR}\frac{q}{R} + \frac{\phi}{c} + \frac{\partial g}{\partial q} = 0\,. \tag{26}$$

From this, follows the expressions for ξ/R and electric potential

$$\frac{\xi}{R} = F_0 f^2 - \frac{R_r^2}{F},\qquad \phi = \phi_0 - f\frac{q}{R}\,, \tag{27}$$

where the designations are introduced

$$f = \frac{1}{c}\frac{\partial g}{\partial m},\quad \phi_0 = -c\frac{\partial g}{\partial q}\,,\quad F = -1 + \frac{2\kappa m}{c^2R} - \frac{\kappa Q^2}{c^4R^2}\,. \tag{28}$$

The resulting solution leads to the metric $\mathbf{M}^{(4)}$:

$$ds^2 = \frac{N^2\left(dx^0\right)^2}{Ff^2 - R_r^2F^{-1}} - \left(Ff^2 - R_r^2F^{-1}\right)\left(dr + N^r dx^0\right)^2 - R^2 d\sigma^2\,. \tag{29}$$

Since time is nowhere explicitly included in the system, we can transform the (5) action from a space-time representation into a configuration one, writing it in a form similar to the Jacobi action [Landau 88]. To do this, from the Lagrangian (5) we get

$$\frac{\partial\mathcal{L}}{\partial N} = -\frac{V^2}{2N^2} + \mathcal{U} = 0 \tag{30}$$

From here we find the multiplier $N = \sqrt{V^2/2\mathcal{U}}$. Excluding N from the action (5), we rewrite it as follows [Barbour 2002, Kiefer 2007, Anderson 2013]

$$S_{tot} = \int dr\int\sqrt{2\mathcal{U}V^2\left(dx^0\right)^2} = \int dr\int\sqrt{D\Omega_{tot}^2} \tag{31}$$

where

$$D\Omega_{tot}^2 = 2\mathcal{U}D\Omega^2 = 2\mathcal{U}\Gamma_{AB}Dq^ADq^B \tag{32}$$

is the supermetric, conformal to the metric of the original CS $D\Omega^2$ (10)-(11).

Note that by transforming [Gladush 21]. the field variables

$$\xi = c\tau - x - \frac{y^2}{R}\,,\quad \varphi = \frac{c^2}{\sqrt{\kappa}}\frac{y}{R}\,,\quad R = c\tau + x \tag{33}$$

the metric $D\Omega^2$ is reduced to the Lorentzian form

$$d\Omega^2 = -c^2D\tau^2 + Dx^2 + Dy^2 = -c^2(d\tau + B^\tau dx^0)^2 + (dx + B^x dx^0)^2 + (dy + B^y dx^0)^2 \tag{34}$$

Thus, the metric $d\Omega^2$ in CS can be considered as the metric of a flat nonholonomic section of a 4-dimensional space. So the structure of the CS is similar to the family of flat nonholonomic sections $\mathbf{M}^{(4)}$. It can be shown that the squared momenta (17) under the transformation (33) also takes the Lorentian form

$$P^2 = -\frac{4\kappa}{c^3}P_RP_\xi + \frac{c}{R^2}P_\phi^2 = -\frac{1}{c^2}P_{\acute{\tau}}^2 + P_{\acute{x}}^2 + (P_{\acute{y}})^2 \tag{35}$$

3. On the quantum geometrodynamics of CBH

The quantum states of the field configuration are determined by the wave functional $\Psi(R,\xi,\phi)$ in the CS. At the same time, the momenta P_A are associated with the momentum operators $\hat{P}_A$, which in the coordinate representation have the form of functional derivatives:

$$\hat{\mathcal{P}}_R = -i\hbar\frac{\delta}{\partial R}, \quad \hat{\mathcal{P}}_\xi = -i\hbar\frac{\delta}{\partial \xi}, \quad \hat{\mathcal{P}}_\phi = -i\hbar\frac{\delta}{\partial \phi} \quad (36)$$

In the case of charge $Q = cP_\phi$ from here we immediately obtain

$$Q \longrightarrow \hat{Q} = c\hat{\mathcal{P}}_\phi = -ic\hbar\frac{\delta}{\partial \phi} \quad (37)$$

Similarly, (14) yields the operator momentum constraint equation

$$R_{,r}\frac{\delta \Psi}{\delta R} - \xi_{,r}\frac{\delta \Psi}{\delta \xi} - 2\xi\frac{\partial}{\partial r}\frac{\delta \Psi}{\delta \xi} = 0 \quad (38)$$

and (15) implies the operator electromagnetic constraint equation

$$\frac{\partial}{\partial r}\left(\frac{\delta \Psi}{\delta \varphi}\right) = 0\,. \quad (39)$$

With the mass function M_{tot} (19) is related to the problem of ordering momentum operators. For the hermiticity of the total mass operator, in the CS with the volume element $dV = (c^{5/2}/2\kappa)Rd\xi dRd\phi$, the following ordering is used $\xi P_\xi \rightarrow \hat{P}_\xi \xi \hat{P}_\xi$. Therefore, the mass function M_{tot} (19) corresponds with the operator

$$\hat{M} = \frac{c^2}{2\kappa}\left(R - \frac{4\kappa^2\hbar^2}{c^6}\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi} - \frac{\kappa\hbar^2}{c^2 R}\frac{\delta^2}{\delta\phi^2} - \frac{R^2}{\xi}\left(R_r\right)^2\right) \quad (40)$$

When the Hamiltonian constraint $H = 0$ (16) is quantized, it is associated with its quantum counterpart $\hat{H}\Psi = 0$, the DeWitt equation. We note that the squared momentum P^2 in (17), as well as the CS metric $d\Omega^2$ can be reduced to the "Lorentzian"form using the transformation (33) . Therefore, in the coordinates $\{\tau, x, y\}$, when quantizing the constraint $H = 0$, you can use the usual quantization recipe in the form (36).

To construct a quantum Hermitian operator in the original curvilinear coordinates $\{R, \xi, \phi\}$, it is necessary to perform an inverse transformation of coordinates and operators, which is equivalent to passing to covariant derivatives $P_A \rightarrow \hat{P}_A = -i\hbar\bigtriangledown_A$ with respect to the metric Γ_{AB} defined in (6). However, in the case under consideration, one should pass to covariant functional derivatives according to the formulas

$$P_A \rightarrow \hat{P}_A = -i\hbar\frac{D}{\delta q_A}\,. \quad (41)$$

Here the covariant functional derivatives are defined as follows

$$\frac{DU_B}{\delta q_A} = \frac{\delta U_B}{\delta q_A} - \Gamma^C_{AB}U_C\,, \quad (42)$$

at that $D\Psi/\delta q_A = \delta\Psi/\delta q_A$. Then, for the momentum squared P^2 (17) in the Hamiltonian constraint (16) after the replacement (41) we have

$$P^2 \longrightarrow \hat{P}^2 = -\hbar^2\Delta \quad (43)$$

Here

$$\Delta = \Gamma^{AB}\frac{D}{\delta q_A}\frac{D}{\delta q_B} = \frac{1}{\sqrt{-\Gamma}}\frac{D}{\delta q_A}\sqrt{-\Gamma}\,\Gamma^{AB}\frac{D}{\delta q_B} = -\frac{2\kappa}{c^4}\frac{\delta^2}{\delta\xi\delta R} - \frac{2\kappa}{c^4}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta}{\delta\xi} + \frac{1}{R^2}\frac{\delta^2}{\delta\varphi^2} \quad (44)$$

is the Laplace-Beltrami operator, which is Hermitian in natural measure. Note that the formula $\bigtriangledown_\xi = \partial/\partial\xi$ takes place, so in the mass operator (2.3) for the momentum operator $\hat{P}_\xi$ it suffices to restrict ourselves to the functional derivative $P_\xi = -i\hbar\delta/\delta\xi$.

As a result, the Hamiltonian constraint (16) leads to the following DeWitt operator

$$\hat{H} = -\frac{c}{2}\hbar^2\Delta - \mathcal{U} \quad (45)$$

or to the DeWitt equation

$$\frac{c}{2}\hbar^2\left(\frac{2\kappa}{c^4}\frac{\delta^2\Psi}{\delta\xi\delta R} + \frac{2\kappa}{c^4}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta\Psi}{\delta\xi} - \frac{1}{R^2}\frac{\delta^2\Psi}{\delta\varphi^2}\right) - \mathcal{U}\Psi = 0\,. \quad (46)$$

States with a certain charge q and mass m are found by solving problems on eigenvalues and eigenfunctions of operators charge Q and mass M

$$\hat{Q}\Psi_q = q\Psi, \quad \hat{M}\Psi_m = m\Psi \quad (47)$$

These equations, taking into account (37) and (40), can be rewritten as follows

$$-ic\hbar\frac{\delta\Psi}{\delta\phi} = q\Psi \quad (48)$$

$$\left(R - \frac{4\kappa^2\hbar^2}{c^6}\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi} - \frac{\kappa\hbar^2}{c^2R}\frac{\delta^2}{\delta\phi^2} - \frac{R^2}{\xi}R_r^2\right)\Psi = \frac{2\kappa m}{c^2}\Psi. \quad (49)$$

By virtue of the relation (48), it follows from the constraint (39) that Ψ does not depend on r. Moreover, (48) implies

$$\Psi\left[R, \xi, \phi; m, q\right] = \psi\left[R, \xi; m, q\right]e^{iq/c\hbar\int\phi(r')dr'}\,. \quad (50)$$

Then, the DeWitt equation (46) becomes

$$\frac{\kappa\hbar^2}{c^3}\frac{\delta^2\Psi}{\delta\xi\delta R} + \frac{\kappa\hbar^2}{c^3}\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta\Psi}{\delta\xi} - \frac{c\hbar^2}{2R^2}\frac{\delta^2\Psi}{\delta\varphi^2} - \mathcal{U}\Psi = 0\,. \quad (51)$$

The equation for the eigenvalues of the mass operator (49) can be rewritten as follows

$$\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi}\psi + \frac{c^6}{4\kappa^2\hbar^2}RF_{tot}\psi = 0\,. \quad (52)$$

The joint solution of the (51) and (52) equations, together with the momentum constraint (38), describes the quantum state of the considered CBH model with fixed charge q and mass m.

4. Geometrodynamics of CBH in the T-region

The T-domain $\mathbf{M}^{(4)}$ CBH is bounded, where the vector Keeling ξ^μ is spatially similar and can be convert to form $\xi^\mu = \delta^\mu_1$. Then the metric (2) can be written as follows

$$ds^2 = \frac{R}{\xi}\left(Ndx^0\right)^2 - \frac{\xi}{R}dr^2 - R^2 d\sigma^2 \qquad (53)$$

In this case, $N^r = 0$, the coordinate system becomes orthogonal and all fields depend only on time. Integration over the coordinate r is replaced by multiplication by the constant $\int dr \to l < \infty$. As a result, the action S_{tot} and the Lagrangian $\mathcal{L}$ (5) take the form

$$S_{tot} \to S = \int L dx^0\,,\ \mathcal{L} \to L = \left(\frac{V^2}{N2} + NU\right). \qquad (54)$$

While, $\mathcal{U} \to U = c^3/2\kappa$ is the potential part of the Lagrangian, $V^2$is the square of the velocity

$$V^2 = \Gamma_{AB}V^AV^B = -\frac{c^3}{\kappa}\xi_{,0}R_{,0} + \frac{1}{c}R^2\,\phi^2_{,0} \qquad (55)$$

Here $V^A = \{V^R = R_{,0},\ V^\xi = \xi_{,0},\ V^\phi = \phi_{,0}\}$ are generalized velocities.

The CS metric is defined similarly to the general case (10):

$$d\Omega^2 = \Gamma_{AB}V^AV^B\left(dx^0\right)^2 = \Gamma_{AB}dq^Adq^B\,, \qquad (56)$$

where the components Γ_{AB} are defined by (6). Then

$$d\Omega^2 = -\frac{c^3}{\kappa}d\xi dR + \frac{1}{c}R^2\,d\phi^2 \qquad (57)$$

Legendre transformation of the system leads to the Hamiltonian action

$$dS = \int(P_\xi d\xi + P_R dR + P_\phi d\phi - NH_0 dx^0) \qquad (58)$$

$$H_0 = \frac{1}{2l}P^2 - lU \ = \frac{c}{2l}\left(-\frac{4\kappa}{c^4}P_RP_\xi + \frac{1}{R^2}\,P^2_\phi + \mu^2\right) \qquad (59)$$

where $H_0 \sim 0$ is the Hamiltonian constraint, $P^2 = \Gamma^{AB}P_AP_B$, $\mu = cl/\sqrt{\kappa}$.

The integrals of system motion are the charge $Q = (c/l)P_\phi$ and mass function

$$M = \frac{1}{2}\left[\frac{c^2}{\kappa}R + \frac{1}{l^2}\left(\frac{4\kappa}{c^4}\xi P^2_\xi + \frac{1}{R}\,P^2_\phi\right)\right]\,, \qquad (60)$$

Together with the Hamiltonian constraint, they, similarly to (21) and (20), lead to momenta

$$P_\xi = \frac{lc^3}{2\kappa}\sqrt{\frac{R}{\xi}F}\,, \qquad (61)$$

$$P_R = \frac{lc^3}{2\kappa}\sqrt{\frac{\xi}{RF}}\left(\frac{\kappa Q^2}{c^4R^2} - 1\right) \qquad (62)$$

where F is defined in (28). Using the integrability conditions for the equations

$$P_R = \frac{\partial S}{\partial R},\ \ P_\xi = \frac{\partial S}{\partial \xi},\ \ P_\phi = \frac{\partial S}{\partial \phi} = \frac{q}{-}\,, \qquad (63)$$

we find the action of S,

$$S = \frac{lc^3}{\kappa}\sqrt{\xi RF} + \frac{lq}{c}\phi + lg(m,q)\,, \qquad (64)$$

which depends on the variables R, ξ and the parameters m, q. From here, and from the relations $\partial S/\partial m = 0$ and $\partial S/\partial q = 0$, we arrive at the motion trajectories in the CS

$$\phi = \phi_0 - f\frac{q}{R}\,, \quad \frac{\xi}{R} = f^2F\,, \qquad (65)$$

where $\phi_0 = -c\partial g/\partial q$, $f = \partial g/c\partial m$. The metric (53) now takes the form

$$ds^2 = \frac{\left(Ndx^0\right)^2}{f^2F} - f^2Fdr^2 - R^2d\sigma^2, \qquad (66)$$

As well as in the general case, we can transform the action (54) from a space-time representation into a CS representation To do this, from the Lagrangian L in (54), we obtain

$$\frac{\partial L}{\partial N} = l\left(-\frac{V^2}{2N^2} + U\right) = 0\,, \qquad (67)$$

which implies $N = V^2/2U$. Substituting N into action (54) we get

$$S = \int l\sqrt{2UV^2}dx^0 = \mu\int\sqrt{cd\Omega^2} \qquad (68)$$

where $d\Omega^2$ received in (57).

Using the transformation (33) of field variables, the metric $d\Omega^2$ CS is reduced to a flat form

$$cd\Omega^2 = -c^2d\tau^2 + dx^2 + dy^2\,. \qquad (69)$$

it that the squared momentum $P^2 = \Gamma^{AB}P_AP_B$ takes the Lorentzian form

$$P^2 = -\frac{4\kappa}{c^3}P_RP_\xi + \frac{c}{R^2}P^2_\phi = -\frac{1}{c^2}P^2_{\acute{\tau}} + P^2_{\acute{x}} + (P_{\acute{y}})^2\,. \qquad (70)$$

As we can see, the corresponding equations of geometrodynamics of the CBH in the T-region are greatly simplified. This is especially important in the case of quantization of the BH. So, the equations system of the quantum theory of BH in functional derivatives for the wave functional $\Psi\left[R,\xi,\phi;m,q\right]$ goes over into the equations system in partial derivatives for the wave function $\Psi(R,\xi,\phi;m,q)$. As a result, we have the DeWitt equation and equations for the eigenvalues ??of mass and charge. The equation for the charge eigenvalue leads to the wave function

$$\Psi[(R,\xi,\phi;m,q)=\psi\left(R,\xi;m,q\right)e^{(iql/c\hbar)\phi} \tag{71}$$

where the function $\psi(R,\xi;m,q)$ obeys the reduced equations DeWitt and the mass eigenvalue

$$\left[\frac{\delta^2}{\delta\xi\delta R}+\frac{1}{R}\frac{\delta}{\delta R}R\frac{\delta}{\delta\xi}+\frac{c^4}{2\kappa\hbar^2}\left(\frac{l^2q^2}{c^2R^2}+\frac{c^2l^2}{\kappa}\right)\right]\Psi=0 \tag{72}$$

$$\frac{\delta}{\delta\xi}\xi\frac{\delta}{\delta\xi}\psi+\frac{c^6}{4\kappa^2\hbar^2}RF\psi=0\,. \tag{73}$$

The joint solution of the system of equations (72) and (73) leads to the following wave function [Gladush 21]

$$\Psi=C\sqrt{\frac{l_{pl}}{R}}J_0\left(\frac{l}{l_{pl}^2}\sqrt{\xi RF_T}\right)e^{\frac{ilq}{\hbar c}\phi}\,, \tag{74}$$

where J_0 is the Bessel function of the first kind of order zero.

We see that in this simplified formulation, the constructed model describes the CBH in the T-region with a continuous spectrum of mass m and charge q.

5. Conclusions

Comparison of the general approach to the geometrodynamics of CBH and the particular approach associated with the reduced model of CBH limited in the T-region of space-time led to the interesting results. The discovered possibility to reduce a nonlinear dynamical system to a linear one, in which all field components are separated, led to the establishment of the configuration space structure as a family of flat non-homogeneous sections of some 4-dimensional space. At the same time, the found transformation led to the construction of the Lorentz form of the momentum square and the subsequent construction of the DeWitt operator containing the Laplace-Beltrami operator in the metric of the configuration space. The construction of this operator and the existence of a solution for the reduced T-model outlines the way to solve the quantum CBH geometrodynamics equations in the general case.

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