DOI:http://dx.doi.org/10.18524/1810-4215.2019.32.182518

SOME COROLLARY FACTS OF THE N-POINT GRAVITATIONAL LENS EQUATION IN A COMPLEX FORM

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ABSTRACT. In the theory of the N-point gravitational lens equation, two groups of problems can be distinguished. These are the so-called primal and inverse problems. Primal problems include problems of image definition in a specified lens for a specified source. Inverse problems include problems of determining a lens, source, or multiple images from one or more specified images. Inverse problem have an important applications.

We studied the equation of the N-point gravitational lens in a complex form. These studies became the basis for the solution of the inverse problem in the following formulation. N-point gravitational lens has specified. It is necessary to determine all other images from one of the images of a point source in N-point gravitational lens. Determine the necessary and sufficient conditions under which this problem has solutions.

The algebraic formulation of the problem has the following form. The equation (of N-point gravitational lens) has specified. It is necessary to solve the problem of solutions unification (to express unequivocally all of the equation solutions through one parameter).

To solve the inverse problem, we used methods of algebraic geometry and function theory. Branches equations of any algebraic function admit unequivocal parameterization by Puiseux series. The solutions of the N-point gravitational lens equation are algebraic functions defined by a certain irreducible polynomial. That polynomial has unequivocally defined by the N-point gravitational lens equation. Thus, the polynomial roots also admits parameterization by Puiseux series.

In simple cases, for lenses with a small number of point masses, the solution can be obtained in a simpler form. In particular, for the Schwarzschild lens and binary lens, the inverse problem has a solution in radicals.

Keywords: gravitational lens, source image, inverse problem, complex analysis.

АНОТАЦІЯ. Серед безлічі проблем і задач, які розглядають в теорії рівняння N-точкових гравітаційних лінз, можна виділити дві групи задач. Це, так звані, прямі і зворотні задачі. До прямих задач відносять задачі визначення зображень в заданій лінзі для заданого джерела. До зворотних – визначення лінзи, джерела або безлічі зображень по одному або декільком заданим зображенням. Зворотна задача має важливе прикладне застосування.

Ми досліджували рівняння N-точкової гравітаційної лінзи в комплексному вигляді. Ці дослідження стали основою для розв'язання оберненої задачі в наступній постановці: задана N – точкова гравітаційна лінза, необхідно, по одному із зображень в ній точкового джерела визначити всі інші зображення цього джерела. Визначити необхідні і достатні умови, при яких ця задача має розв'язок.

В алгебраїчній формулюванні задача має вигляд. Задано рівняння N-точкової гравітаційної лінзи. Необхідно розв'язати задачу уніформізації коренів (однозначно виразити усі корені рівняння через один параметр).

Для розв'язання оберненої задачі, ми використовували методи алгебраїчної геометрії і теорії функцій. Рівняння гілок будь-якої алгебраїчної функції допускають однозначну параметризацію рядами Бюрмана–Лагранжа. Корені рівняння N-точкової гравітаційної лінзи є алгебраїчними функціями визначеними незведеним многочленом. Цей многочлен однозначно визначається рівнянням N-точкової гравітаційної лінзи. Отже, корені цього многочлена, також допускають параметризацію рядами Бюрмана–Лагранжа.

У простих випадках, для лінз з малим числом точкових мас, розв'язок може бути отриманий в більш простому вигляді. Зокрема, для лінзи Шварцшильда і бінарної лінзи зворотна задача має розв'язок в радикалах.

Ключові слова: гравітаційні лінзи, зображення джерела, обернена задача комплексний аналіз.

1. Introduction

By methods of algebraic geometry we studied the equation of N-point gravitational lens. This has led us to a special case solution of one of the inverse problems. Expressly, N-point gravitational lens is specified and the coordinates of one of the image of the point source in it are known. It is necessary to determine the coordinates of all other images. Formulate the necessary and sufficient conditions under which this problem has solutions.

The algebraic formulation of the problem has the following form. The equation (of N-point gravitational lens) has specified. It is necessary to solve the problem of solutions unification (to express unequivocally all of the equation solutions through one parameter).

In this paper, we used methods of algebraic geometry and function theory. The solutions of the equation N-point gravitational lens possible to parameterize by Puiseux series.

For lenses with a small number of point masses, the solution can be obtained in a simpler form. In particular, for the Schwarzschild lens and binary lens, the inverse problem has a solution in radicals.

2. N-point gravitational lens equation in complex form

N-point gravitational lens equation can be written in complex form:

$$\zeta = z - \overline{w(z)}, \tag{1}$$

wherein

$$w = \sum_{n=1}^{N} m_n \frac{1}{(z - A_n)}; \sum_{n=1}^{N} m_n = 1, \tag{2}$$

where m_n - normalized point masses included in the lens, A_n – their complex coordinates [Kotvytskiy, SHablenko, Bronza, 2018; Witt, 1990]. In [Dank, Heyrovský, 2015], it has also proved that:

$$w = \frac{1}{\deg P(z)} \frac{P'(z)}{P(z)}, \tag{3}$$

where $P(z) = \prod_{n=1}^{N} (z - A_n)^{m_n}$.

Using equation (1), new proofs have been obtained of previously known theorems about images of a point source in the N-point gravitational lens:

- about single extended image (Einstein ring) [Kotvytskiy et al, 2017];
- about infimum of a number of point source images in the N-point gravitational lens [Dank & Heyrovský, 2015];
- about supremum of a number of point source images in the N-point gravitational lens [Osmayev & Matvienko, 2018].

Using equation (1), a new, previously unknown, result has also been obtained. The problem of unequivocal parameterization of image coordinates has been solved. Algorithms that admit to express unequivocal the coordinate of any image through the same parameter have been designed. In particular, the coordinate of one of the images might be used as a parameter.

3.Solution of the problem of image coordinates parameterization

For the Schwarzschild lens, a theorem holds.

Theorem 1. Let z be the complex coordinate of one of the images in the Schwarzschild lens, then the coordinate of the second image is $-\frac{1}{\bar{z}}$.

Proof. Images in a Schwarzschild lens with coordinates z and $-\frac{1}{\bar{z}}$ are images of the same source as

$$\zeta(z) = \zeta(-\frac{1}{\bar{z}}) \tag{4}$$

Indeed, equation (1) for the Schwarzschild lens in complex form has the form:

$$\zeta = z - \frac{1}{\bar{z}}, \tag{5}$$

if mass of the lens is located at the origin of the coordinates.

Substitute in equation (5) $-\frac{1}{\bar{z}}$, we have:

$$\zeta(-\frac{1}{\bar{z}}) = -\frac{1}{\bar{z}} - \frac{1}{\overline{(-\frac{1}{\bar{z}})}} = -\frac{1}{\bar{z}} - \frac{1}{(-\frac{1}{z})} =$$
$$= -\frac{1}{\bar{z}} + z = \zeta(z) \implies \zeta(z) = \zeta(-\frac{1}{\bar{z}}).$$

Therefore, relation (4) holds.

The theorem is proved.

Each point source, which located not at the origin of the coordinates, has exactly two point images in the Schwarzschild lens. Therefore, the problem of image coordinates parameterization for the Schwarzschild lens is solved.

Corollary to Theorem 1. Let g_1, g_2 the affine coordinates of one of the images in the Schwarzschild lens, then the affine coordinates of the second image

$$-g_1(g_1^2 + g_2^2), -g_2(g_1^2 + g_2^2)^{-1}.$$

Proof. Let $z = g_1 + g_2 i$, then

$\mathrm{Re}(-\frac{1}{z}) = -g_1(g_1^2 + g_2^2), \mathrm{Im}(-\frac{1}{z}) = -g_2(g_1^2 + g_2^2),$

The corollary is proved.

For polynomials $P(z)$ in one variable over z the field of complex numbers, the following theorem holds.

Theorem 2. Let $P(z)$ the polynomial over the field of complex numbers and $2 \le \deg P(z) \le 5$, then, the problems of parameterization of roots of the polynomial $P(z)$ are solvable in radicals.

Proof. Polynomial remainder theorem implies that the difference $P(z) - P(t)$ is divided exactly by the binomial $(z - t)$. Then

$$Q(z,t) = \frac{P(z) - P(t)}{z - t}, \tag{6}$$

is a polynomial in two variables. The degree of the polynomial $Q(z,t)$ for each of the variables

$$\deg_z(Q(z,t)) = \deg_t(Q(z,t)) = n - 1, \tag{7}$$

where $n = \deg P(z)$.

If $n \le 5$, then the polynomial $Q(z,t)$, as a polynomial from the variable z, over the field of rational functions from t, has the degree $\deg_z(Q(z,t)) \le 4$. Therefore, each of its $n-1$ roots can be expressed in radicals through its coefficients. Since the coefficients of the polynomial $Q(z,t)$ are rational functions of t, its roots are unequivocally expressed in terms of t. The variable t will be considered as a parameter.

Thus, the problem of parameterization of the roots of a polynomial $Q(z,t)$ is solvable in radicals.

We have an expression for $n-1$ roots of polynomial $P(z)$ through t. We will put the remaining root of polynomial $P(z)$ equal to t.

The theorem is proved.

Remark of the theorem 2. Let the polynomial $P(z)$ meet the conditions of theorem 2, then $n-1$ roots of polynomial $P(z)$ can be expressed in terms of the remaining root. Indeed, let have t equal to one of the roots of the polynomial.

Example to theorem 2. Let the equation be specified

$$z^3 + az^2 + bz + c = 0 \tag{8}$$

It is necessary to express two roots z_1, z_2 of equation (8) by the third z_3.

Solution. Denote the polynomial standing in the left side of equation (8) by $P_3(z)$, i.e.

$$P_3(z) = z^3 + az^2 + bz + c. \tag{9}$$

The difference $P_3(z) - P_3(t)$ is divided exactly by the binomial $(z - t)$:

$$\begin{aligned} &P_3(z) - P_3(t) = z^3 + az^2 + bz + c- \\ &-(t^3 + at^2 + bt + c) = \\ &= z^3 - t^3 + a(z^2 - t^2) + b(z - t) = \\ &= (z - t)(z^2 + zt + t^2 + az + at + b) \end{aligned} \tag{10}$$

If z the root of a polynomial $P_3(z)$, then from (10)

$$z^2 + zt + t^2 + az + at + b = 0 \tag{11}$$

Equation (11) as regard to the variable is the quadratic equation:

$$z^2 + (t + a)z + t^2 + at + b = 0 \tag{12}$$

The roots of equation (12) are equal to:

$$z_{1,2} = \frac{t + a \pm \sqrt{(t + a)^2 - 4(t^2 + at + b)}}{2}.$$

With the $z_3 = t$ we have:

$$z_{1,2} = \frac{z_3 + a \pm \sqrt{-4z_3^2 - 2az_3 - 4b + a^2}}{2}.$$

The roots of equation (8) z_1, z_2 are expressed by the third root z_3.

In the general case, for the N-point gravitational lens we have:

From equation (1), we have:

$$\zeta = z - \sum_{n=1}^{N} \frac{m_n}{\overline{z} - \overline{A_n}}. \tag{13}$$

We proceed to the complex conjugation in both parts of the equation (13):

$$\overline{\zeta} = \overline{z} - \sum_{n=1}^{N} \frac{m_n}{z - A_n}. \tag{14}$$

Substitute the variable $\overline{z}$ from equation (14 to (13), we have:

$$\zeta = z - \sum_{n=1}^{N} \frac{m_n}{\overline{z} - \sum_{n=1}^{N} \frac{m_n}{z - A_n} + \overline{\zeta} - \overline{A_n}}. \tag{15}$$

Equation (15) is brought into polynomial form and all summands are transferred to the left side of the equation. Denote the left side of the obtained equation by $F(z, \zeta, \overline{\zeta})$.

Compute the expression

$$Q(z,t) = \frac{F(z, \zeta, \overline{\zeta}) - F(t, \zeta, \overline{\zeta})}{z - t} = \frac{F(z) - F(t)}{z - t}. \tag{16}$$

The following theorem holds.

Theorem 3. The expression $\frac{F(z)-F(t)}{z-t}$ is a polynomial in two variables z and t.

Proof. The validity of the theorem follows from polynomial remainder theorem. The residue from dividing the polynomial $F(z)$ by the difference $(z - t)$ is equal to $F(t)$. And from the corollary of polynomial remainder theorem. The number t is the root of the polynomial $F(z)$ if and only if the expression

$F(z) - F(t)$ is divided without residue by the binomial $(z - t)$. The theorem is proved.

Polynomial $Q(z,t)$ will be termed a difference polynomial for polynomial $F(z)$. Polynomial $Q(z,t)$ as regard to the variable z has the degree [Kotvytskiy, Bronza, Shablenko, 2017]:

$$\deg_z Q(z,t) = \deg_z F(z,\zeta,\overline{\zeta}) - 1 \leq \\ \leq N^2 + 1 - 1 = N^2 \qquad (17)$$

The Z polynomial, actually, is reduced. Polynomial is presented in the form of a product of irreducible factors [Bronza & Kotvytskiy, 2017; Bronza, 2016], if it is reduced.

The problem of root parameterization of a polynomial $Q(z,t)$ may be solved by using Puiseux series. For this purpose, there is a geometric method known as "Newton's Diagram", see [Chebotarev, 1948].

4. Solution of inverse problem for binary symmetric lens

Let $S_2 = S_2(-a, a)$ is a binary symmetrical lens with masses $m_1 = m_2 = 0.5$, which located at the $-a$ and a points on the real axis. The lens equation has the form:

$$\zeta = z - \frac{\overline{z}}{\overline{z}^2 - a^2}. \qquad (18)$$

Exclude from equation (18) and equation of complex-conjugate to it $\overline{z}$, all summands are transferred to the left side and factorized, we have:

$$(z-\zeta)[(\overline{\zeta}^2 - a^2)(z^2-a^2)^2 + 2\overline{\zeta}(z^2-a^2)z + \\ z^2] - (\overline{\zeta}(z^2-a^2) + z)(z^2-a^2) = 0. \qquad (19)$$

Denote the left side of equation (19) by $F(z,\zeta,\overline{\zeta})$.

Compute the polynomial by the formula (16).

$$Q = (-a^2 + \overline{\zeta}^2)z^4 + [(-a^2+\overline{\zeta}^2)t + \\ + (a^2\zeta + \overline{\zeta} - \zeta\overline{\zeta}^2)]z^3 + [(-a^2+\overline{\zeta}^2)t^2 + \\ + (a^2\zeta + \overline{\zeta} - \zeta\overline{\zeta}^2)t + (2a^4 - 2\zeta\overline{\zeta} - 2a^2\overline{\zeta}^2)]z^2 + \\ + [(-a^2+\overline{\zeta}^2)t^3 + (a^2\zeta + \overline{\zeta} - \zeta\overline{\zeta}^2)t^2 + \\ + (2a^4 - 2\zeta\overline{\zeta}^2 - 2a^2\overline{\zeta}^2)t + (-2a^4\zeta + 2a^2\zeta\overline{\zeta}^2 - \zeta)]z + \\ + (-a^2+\overline{\zeta}^2)t^4 + (a^2\zeta + \overline{\zeta} - \zeta\overline{\zeta}^2)t^3 + \\ + (2a^4 - 2\zeta\overline{\zeta}^2 - 2a^2\overline{\zeta}^2)t^2 + \\ + (-2a^4\zeta + 2a^2\zeta\overline{\zeta}^2 - \zeta)t + (-a^6 + a^4\overline{\zeta}^2 + a^2 + 2a^2\zeta\overline{\zeta}) \qquad (20)$$

Polynomial $Q(z,t)$ as regard to the variable z has the fourth power degree. Its coefficients are polynomials in the variable t. $Q(z,t)$ roots may be expressed in radicals from its coefficients. Since, as the coefficients $Q(z,t)$, there are polynomials in the variable t, we have: $z_{1,2,3,4} = z(t)$. Among other things, since the polynomial $F(z,\zeta,\overline{\zeta})$ has another root $z_5 = t$, we have:

$$z_{1,2,3,4} = z(z_5). \qquad (21)$$

The formula (21) proved the possibility to express all roots of the polynomial $Q(z,t)$ by one parameter t.

Thus, the inverse problem for a binary lens is solvable in radicals.

In particular, if a point source is located on the real axis, the lens has three images on the real axis, at the points with coordinates $z_i, i = 1,2,3$, the relation holds:

$$z_{2,3} = \frac{z_1 \pm z_1\sqrt{1 + 4(z_1^2 - a^2)(z_1^2 - a^2 + 1)}}{4(z_1^2 - a^2)}. \qquad (22)$$

In total, a source in the binary symmetrical lens may have either 3 or 5 images. For any real Z source is located on the real axis, it also has either 3 or 5 images. Both cases are realized.

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