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CLASSICAL DESCRIPTIONS OF THE GEOMETRODYNAMICS OF CHARGED BLACK HOLES

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ABSTRACT. Analytical aspects of the classical geometrodynamics of charged black holes are considered. The classical model of the charged BHs is the spherically symmetric configuration of the electromagnetic and gravitational fields in GR. The feature of such dynamic systems is that, in addition to the Killing vector, they admit two motion integrals: the total mass M and the charge Q of the configuration. Using these conservation laws, as well as the Hamiltonian constraint, the momenta as functions of configuration variables are found. In addition, the integrability conditions for the momenta as functional derivatives of the action are satisfied. This allows us to calculate the functional of action, which is a solution of the Einstein-Hamilton-Jacobi equation. Variations of action functional with respect to the mass M and charge Q lead to the solution of the Einstein equations in the configuration space.

Keywords: electromagnetic and gravitational fields, geometrodynamics, constraints, configuration space, Einstein-Hamilton-Jacobi equation.

АНОТАЦІЯ. Розглянуто аналітичні аспекти класичної геометродинаміки для сферично-симетричної (СС) конфігурації електромагнітного і гравітаційного полів ЗТВ. Особливістю цих конфігурацій є те, що вони допускають два інтеграла руху – загальну масу і заряд. Дія Ейнштейна-Гільберта для конфігурації після вимірної редукції за допомогою перетворення Лежандра зводиться до гамільтонової дії. Використовуючи закони збереження маси і заряду, а також гамільтонову в'яз, знайдені імпульси як функції змінних конфігурації. Крім того, виконуються умови інтегрованості імпульсів як функціональних похідних дії. Це дозволяє нам отримати функціонал дії як розвє'язок рівняння Ейнштейна-Гамільтона-Якобі в функціональних похідних. Варіації функціоналу дії щодо маси M і заряду Q конфігурації призводять до траєкторіях руху в конфігураційному просторы.

Ключові слова: електромагнітне і гравітаційне поля, геометродинаміка, в'язи, конфігураційний простір, рівняння Ейнштейна-Гамільтона-Якобі.

1. Introduction

As is known, the space-time metric M^4 for a spherically symmetric (SS) configuration of the electromagnetic and gravitational fields in GR (as well as with the cosmological constant) admits the Killing vector. Therefore, in the R-region, when choosing Killing time, the fields do not have dynamic degrees of freedom. Therefore, to study the questions of quantization, in [Gladush 2016, 2018], we limited ourselves to considering the T-region, where these fields have a dynamic meaning. The limited nature of this direction dictates us to consider a more general, geometrodynamics approach to the SS configuration of electromagnetic and gravitational fields [Kuchar 1994, Louko 1996, Makela 1998]. In this approach, a 3 + 1-splitting of M^4 into a one-parameter family of space-like hypersurfaces is constructed. The corresponding parameter labeling the hypersurfaces determines the time coordinate of a normal reference frame and describes the evolution of geometric quantities defined on these hypersurfaces. This introduces the dynamics of objects defined on these hypersurfaces in M^4.

2. The basic dynamic values of the configuration

We start from the general Arnowitt-Deser-Misner SS line element.

$$ds^2 = N^2 \left(dx^0\right)^2 - L^2 \left(dr + N^r dx^0\right)^2 - R^2 d\sigma^2 \,, \quad (1)$$

where $x^0 = ct$ and $d\sigma^2 = d\theta^2 + \sin^2\theta d\alpha^2$. In this formula the lapse N and the shift N^r, as well as the quantities L and R, which are considered as the dynamical variables of the spacetime geometry, are assumed to be functions of the time coordinate $x^0 = ct$ and the radial coordinate r only. The electromagnetic potential is taken to be described by the SS one-form

$$A = A_0 dx^0 + A_r dr = \varphi dx^0 + \phi dr \,, \quad (2)$$

where $A_0 = \varphi\left(x^0, r\right)$ and $A_r = \phi\left(x^0, r\right)$. For the

Einstein-Maxwell theory the action is

$$S = -\frac{1}{16\pi c}\int_{M^{(4)}}\left(\frac{c^4}{\kappa}{}^{(4)}R + F^{\mu\nu}F_{\mu\nu}\right)\sqrt{-g}d^4x \quad (3)$$
$$+(\text{boundary terms})\,,$$

where $g = -N^2L^2R^4\sin^2\theta$ is the determinant of the metric (1), ${}^{(4)}R$ is the Ricci scalar, and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor. Inserting the SS fields (1) and (2) and integrating over the angles α and θ we obtain, up to boundary terms, the action

$$S = \int \mathcal{L}d^2x\,, \quad (4)$$

$$\mathcal{L} = \frac{1}{N}\mathcal{T} + \frac{c^3}{2\kappa}NU\,. \quad (5)$$

Here $\mathcal{L}$ the Lagrangian of system, $\mathcal{T}$ and U are the kinetic and potential parts of system:

$$\mathcal{T} = -\frac{c^3}{2\kappa}\left[2R\left(L_{,0} - (LN^r)_{,r}\right)\right.$$
$$\left.+L\left(R_{,0} - N^rR_{,r}\right)\right]\left(R_{,0} - N^rR_{,r}\right) \quad (6)$$
$$+\frac{1}{2c}\frac{R^2}{L}E^2\,,$$

$$U = L - \frac{1}{L}R_r^2 + 2\frac{R}{L^2}R_rL_r - 2\frac{R}{L}R_{rr}\,, \quad (7)$$

where

$$E = F_{0r} = -F_{r0} = \phi_{,0} - \varphi_{,r}\,.$$

We give the constraints that arise from the Lagrangian (5). First we write down the primary constraints

$$P_N = \frac{\partial\mathcal{L}}{\partial N_{,0}} = 0, \qquad P_{N^r} = \frac{\partial\mathcal{L}}{\partial N^r_{,0}} = 0,$$

$$P_\varphi = \frac{\partial\mathcal{L}}{\partial\varphi_{,0}} = 0\,.$$

Next, we write down the secondary constraints, are the Hamiltonian, momentum (diffeomorphism) and Gaussian ones, respectively

$$\frac{\partial\mathcal{L}}{\partial N} = -\frac{1}{N^2}\mathcal{T} + \frac{c^3}{2\kappa}U =$$
$$-\frac{1}{N^2}\left\{-\frac{c^3}{2\kappa}2R\left(L_{,0} - (LN^r)_{,r}\right)\left(R_{,0} - N^rR_{,r}\right)\right.$$
$$\left.-\frac{c^3}{2\kappa}L\left(R_{,0} - N^rR_{,r}\right)^2 + \frac{1}{2c}\frac{R^2}{L}E^2\right\} \quad (8)$$
$$+\frac{c^3}{2\kappa}\left[L - \frac{1}{L}R_r^2 + 2\frac{R}{L^2}R_rL_r - 2\frac{R}{L}R_{rr}\right] = 0\,,$$

$$\frac{\delta\mathcal{L}}{\delta N^r} = \frac{c^3}{\kappa}\frac{R}{N}\left\{\left(L_{,0} - \frac{V}{N}\left(N^rL\right)_{,r}\right)R_{,r}\right. \quad (9)$$
$$\left.-L\left(R_{,0r} - N^rR_{,rr} - \frac{N_{,r}}{N}R_{,0}\right)\right\} = 0\,,$$

$$\frac{\delta\mathcal{L}}{\delta\varphi} = \frac{1}{c}\frac{\partial}{\partial r}\left(\frac{R^2}{NL}E\right) = 0\,. \quad (10)$$

The last constraint has the solution

$$\frac{R^2}{NL}E = Q\,. \quad (11)$$

It follows from here that

$$E = \phi_{,0} - \varphi_{,r} = NL\frac{Q}{R^2}\,. \quad (12)$$

This determines the electric field strength E of the charge Q.

The canonical momenta, which are conjugate to the variables L, R and α are

$$P_R = \frac{\partial\mathcal{L}}{\partial R_{,0}} = -\frac{c^3}{\kappa N}\left[L\left(R_0 - N^rR_r\right)\right.$$
$$\left.+R(L_{,0} - (LN^r)_{,r})\right]\,, \quad (13)$$

$$P_L = \frac{\partial\mathcal{L}}{\partial L_{,0}} = -\frac{c^3}{\kappa}\frac{R}{N}\left(R_0 - N^rR_r\right)\,, \quad (14)$$

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\phi_{,0}} = \frac{R^2}{cNL}E = \frac{Q}{c}\,. \quad (15)$$

A Legendre transformation, up to surface terms, leads to the Hamiltonian action

$$S_\Sigma[L, R, \phi, P_L, P_R, P_\phi; N, N^r, \varphi]$$
$$= \int dx^0\int dr\left(P_RR_{,0} + P_LL_{,0} + P_\phi\phi_{,0}\right. \quad (16)$$
$$\left.-NH - N^rH_r - \varphi H_\varphi\right)\,.$$

where

$$H = -\frac{\partial\mathcal{L}}{\partial N} = \frac{\kappa}{c^3}\left(\frac{L}{2R^2}P_L^2 - \frac{1}{R}P_RP_L\right) + \frac{cL}{2R^2}P_\phi^2$$
$$-\frac{c^3}{\kappa}\left(\frac{L}{2} - \frac{1}{2L}R_{,r}^2 + \frac{R}{L^2}R_{,r}L_{,r} - \frac{R}{L}R_{,rr}\right) = 0\,, \quad (17)$$

$$H_r = -\frac{\delta\mathcal{L}}{\delta N^r} = L\left(P_L\right)_{,r} - P_RR_r = 0\,, \quad (18)$$

$$H_\varphi = -\frac{\delta\mathcal{L}}{\delta\varphi} = -\frac{\partial}{\partial r}P_\phi = 0\,. \quad (19)$$

Here the Hamiltonian, diffeomorphism and Gauss law constraints, which are expressed through momenta

3. The mass function and momenta of field configuration

The Einstein equations for the configuration under consideration lead to the conservation laws of charge Q (11) and the total mass function M (Gladush 2012).

$$M_{tot} = \frac{c^2}{2\kappa}R\left(1 + (\nabla R)^2\right) + \frac{Q^2}{2c^2R}\,. \quad (20)$$

For metric (1), the mass function is defined as follows

$$M_{tot} = \frac{c^2}{2\kappa}R\left[1 + \frac{1}{N^2}\left(R_{,0} - N^rR_{,r}\right)^2 - \frac{1}{L^2}R_{,r}^2\right]$$
$$+ \frac{R^3}{2c^2N^2}\frac{E^2}{L^2} = m = const\,, \quad (21)$$

or, by the formulae (13,14,15) one gets M in terms of momenta

$$M_{tot} = \frac{c^2}{2\kappa}\left[R+\left(\frac{\kappa}{c^3}\right)^2\frac{(P_L)^2}{R}-\frac{R}{L^2}R_{,r}^2\right] + \frac{P_\phi^2}{2R}=m\,. \quad (22)$$

Using the mass function and the Hamiltonian constraint (17), one can find the momenta P_L and P_R on the CS, which allows one to construct the action of the system as a solution of the Einstein-Hamilton-Jacobi equation (EHJ). Indeed, taking into account (15), from (22) we find P_L as a function of mass m, charge Q, and configuration variables of the system

$$P_L=\frac{c^3}{\kappa}R\sqrt{F_{tot}}\,, \quad (23)$$

where

$$F_{tot}=\frac{R_{,r}^2}{L^2}+F\,, \quad (24)$$

$$F=-1+\frac{2\kappa m}{c^2R}-\frac{\kappa Q^2}{c^4R^2}\,. \quad (25)$$

If one substitutes this expression for P_L into Eq. (17), we get the momentum P_R

$$P_R=\frac{c^3}{\kappa\sqrt{F_{tot}}}\left[\left(\frac{R}{L}R_{,r}\right)_{,r}+\left(\frac{\kappa m}{c^2R}-1\right)L\right]\,. \quad (26)$$

It is easy to check on that the momentum constraint (18) for the momenta (23), (26), i.e., the condition for the invariance of the action functional, is satisfied identically.

4. The action and the system trajectories in the configuration space

The EHJ equation for the action $S[L,R,\phi;Q,m;r]$ can be obtained by substituting the functional derivatives

$$P_L=\frac{\delta S}{\delta L},\ P_R=\frac{\delta S}{\delta R},\ P_\phi=\frac{\delta S}{\delta\phi}=\frac{Q}{c} \quad (27)$$

into constraint (17).

However, we will find the action in a simpler way. The last equation in (27) gives

$$S[L,R,\phi;Q,m;r]=S_0[L,R;Q,m;r]+\int\frac{Q}{c}\phi dr\,, \quad (28)$$

where $S_0[L,R;Q,m;r]$ is a functional independent of ϕ, and moreover,

$$P_L=\frac{\delta S_0}{\delta L},\quad P_R=\frac{\delta S_0}{\delta R}\,. \quad (29)$$

The expression for P_L does not contain the derivatives of L, therefore, following (Louis-Martinez 1994), the first equation in (29) is directly integrated:

$$S_0[L,R;Q,m;r] = \frac{c^3}{\kappa}\int LRdr\left\{\sqrt{F_{tot}}-\frac{R_{,r}}{2L}\ln\frac{L\sqrt{F_{tot}}+R_{,r}}{L\sqrt{F_{tot}}-R_{,r}}\right\} + G\left(R;M,Q;r\right) \quad (30)$$

or

$$S_0[L,R;Q,m;r]S = \frac{c^3}{\kappa}\int LRdr\left\{\sqrt{F_{tot}}-\frac{R_{,r}}{L}arcth\frac{R_{,r}}{L\sqrt{F_{tot}}}\right\} + G\left(R;M,Q;r\right)\,, \quad (31)$$

where $G\left(R;M,Q;r\right)$ is a functional independent of L. Using the second equation in (29), from (30) we obtain

$$P_R=\frac{\delta S_0}{\delta R}=P_R+\frac{\delta S}{\delta R}G\left(R;m,Q;r\right)\,. \quad (32)$$

This implies that $G\left(R;m,Q;r\right)=G\left(m,Q;r\right)$ is a functional independent of R. Here we indirectly checked the integrability conditions of functional equations (29)

As a result, we come to the action functional in the CS

$$S[L,R,\phi;Q,m;r] = \frac{c^3}{\kappa}\int LRdr\left\{\sqrt{F_{tot}}-\frac{R_{,r}}{2L}\ln\frac{L\sqrt{F_{tot}}+R_{,r}}{L\sqrt{F_{tot}}-R_{,r}}\right\} + \int\frac{Q}{c}\phi dr+\int drg\left(m,Q;r\right)\,, \quad (33)$$

as a solution of the EHJ equation. Here $g\left(M,Q;r\right)$ is an arbitrary function of M,Q and r.

The system trajectories in the CS or the solution of Einstein equations follow from the relations

$$\frac{\delta S}{\delta m}=c\frac{L}{F}\sqrt{F_{tot}}-cf\left(r\right)=0\,, \quad (34)$$

$$\frac{\delta S}{\delta Q}=-L\frac{Q\sqrt{F_{tot}}}{cRF}+\frac{1}{c}\phi-\frac{1}{c}\phi_{(0)}=0\,, \quad (35)$$

where the following notation is introduced

$$f\left(r\right)=-\frac{\partial g\left(m,Q;r\right)}{c\partial m}\,, \quad (36)$$

$$\phi_{(0)}=-c\frac{\partial g\left(m,Q;r\right)}{\partial Q}\,. \quad (37)$$

Hence it can be seen, that

$$L=\sqrt{Ff^2\left(r\right)-\frac{R_{,r}^2}{F}}\,, \quad (38)$$

$$\phi=\phi_{(0)}+f\left(r\right)\frac{Q}{R}\,. \quad (39)$$

The obtained relations determine the dependence of the dynamic variables L and ϕ on the coordinate r, the variable $R(r)$ and its derivative with respect to r.

From here we find the space-time metric $\mathbf{M}^4$ and the potential of the electromagnetic field

$$ds^2 = N^2\left(dx^0\right)^2 - \quad (40)$$
$$-\left(Ff^2(r) - F^{-1}R_{,r}^2\right)\left(dr + N^r dx^0\right)^2 - R^2 d\sigma^2\,,$$

$$A = \varphi dx^0 + \left(\phi_{(0)} + f\left(r\right)\frac{Q}{R}\right)dr\,. \quad (41)$$

The lapse N and the shift N^r can be found by passing to the Reissner-Nordstrom solution in the T-region or by using the corresponding time recovery procedure.

5. Minisuperspace metric

From relation (8) we have $N = \sqrt{2\kappa\mathcal{T}/c^3U}$). Therefore, the action (4) can be rewritten as follows

$$\begin{aligned} S &= \int dx^0 \int dr\left(\frac{1}{N}\mathcal{T} + \frac{c^3}{2\kappa}NU\right) \\ &= 2\int dx^0 \int dr\sqrt{\frac{c^3}{2\kappa}\mathcal{T}U} \end{aligned} \quad (42)$$

or, taking into account (6), in the form

$$S = \sqrt{\frac{2c^3}{\kappa}}\int dr \int D\Omega \quad (43)$$

Here we introduce the Lie differentials by the formulas

$$\begin{aligned} DL &= \left(L_{,0} - \left(LN^r\right)_{,r}\right)dx^0 \quad (44)\\ &= dL - \left(LN^r\right)_{,r}dx^0\,, \\ DR &= \left(R_{,0} - N^r R_{,r}\right)dx^0 \quad (45)\\ &= dR - N^r R_{,r}dx^0\,, \\ D\phi &= \left(\phi_{,0} - \varphi_{,r}.\right)dx^0 = d\phi - \varphi_{,r}.dx^0\,. \quad (46) \end{aligned}$$

Then, the action S can be rewritten as

$$S = \sqrt{\frac{2c^3}{\kappa}}\int dr \int D\Omega\,, \quad (47)$$

where $D\Omega^2$ is the supermetric CS

$$D\Omega^2 = Ud\Omega_0^2 =$$
$$= \tfrac{1}{2c}U\left[-\tfrac{c^4}{\kappa}\left(2RDLDR + LDR^2\right) + \tfrac{R^2}{L}D\phi^2\right]. \quad (48)$$

In the simplest case of curvature coordinates in the T-region, when $R = cT(x^0)$, and all spatial derivatives disappear, we have $U = L$ and the action takes the form

$$S_\Sigma = \quad (49)$$
$$= \tfrac{c^3}{\kappa}\int dr \int \sqrt{-\left(2RdR + LdR\right)LdL + \tfrac{\kappa}{c^4}R^2\,d\phi^2}.$$

Using the field transformation

$$L^2 = \frac{1}{c\tau + x}\left(c\tau - x - \frac{y^2}{c\tau + x}\right), \quad (50)$$

$$\phi = \frac{c^2}{\sqrt{\kappa}}\frac{y}{c\tau + x}\,, \quad (51)$$

$$R = \frac{\sqrt{\kappa}}{c^2}\left(c\tau + x\right), \quad (52)$$

supermetric of $D\Omega^2$ is reduced to the Lorentz form, i.e. CS is flat

$$d\Omega^2 = L\Omega_0^2 = -c^2 d\tilde{\tau}^2 + d\tilde{x}^2 + dy^2\,, \quad (53)$$

It follows that minisupermetrics admits the motions group $O(1,2)$. Note that the supermetric kinetic part

$$d\Omega_0^2 = \mathcal{T}\left(dx^0\right)^2$$
$$= \tfrac{1}{2c}\left[-\tfrac{c^4}{\kappa}2RDLDR + LDR^2 + \tfrac{R^2}{L}D\phi^2\right], \quad (54)$$

is conformally flat.

6. Conclusions

It is to be noted that the inequality $F_{tot} = R_{,r}^2/L^2 + F > F > 0$ in equation (23) determines the classically admissible region. Its boundary is given by the equation roots $F_{tot} = 0$. On the other hand, the T-region of Reissner-Nordstrom space-time is specified by the condition $F > 0$, while the horizon is determined by the equation $F = 0$. It is easy to see that the T-region is contained in a classically admissible region.

It is also interesting to note that the complete integrability of the considered system is preserved when the cosmological constant is added. The complete integrability of the system is due to the fact that the considered field configuration in the SS case does not have local degrees of freedom. However, the inclusion of a scalar field in the configuration completely destroys the picture. In this case, the mass function is not conserved, and the corresponding equations in the general case cannot be analytically solved. Physically, this is due to the fact that spherical symmetry allows scalar field waves. In this case, the configuration turns out to be a dynamic system with an infinite number of freedom degrees.

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