SUPERSPACE APPROACH TO THE QUANTIZATION OF CHARGED BLACK HOLES WITH ALLOWANCE FOR THE COSMOLOGICAL CONSTANT

M.G. Holovko, V.D. Gladush

Dnipro National University, Gagarin Ave, 725014, Ukraine,
dilmendar@gmail.com, vgladush@gmail.com

ABSTRACT. In the present paper, we study the geometry of a mini-superspace and its relation to the corresponding space-time geometry of a spherically symmetric configuration of electromagnetic and gravitational fields, taking into account the cosmological constant, and the construction of the wave function of a quantum system. By the generalized Birkhoff theorem, for this configuration we can introduce the R- and T-regions, which simplifies the description of the dynamical system. Proceeding from the standard classical Einstein-Hilbert action, a Lagrangian of the fields configuration is constructed for a spherically symmetric space-time. The Lagrangian of the system is degenerate and contains a non-dynamic degree of freedom, which leads to a constraint. After eliminating the constraints, we proceed to the description of the dynamic system in the configuration space (minisuperspace). We consider additional conserved physical quantities: the total mass and the charge of the system. We note that the geometry of the minisuperspace turns out to be conformally flat. In addition to the standard horizons inherent in a charged black hole, space-time has an additional cosmological horizon. In the configuration space the simplest invariants of the curvature tensor: the scalar curvature, the square of the Ricci tensor, the Kretschmann invariant, are vanish, while the components of the Ricci tensor and the curvature tensor diverge on the minisuperspace analogue of the cosmological horizon.

Within the framework of canonical quantum gravity with material sources, physical states are found by solving the Hamiltonian constraint in the operator form for the wave function of the system defined on the minisuperspace, taking into account conserved additional quantities. Formal quantization in the R-region can be regarded as an analytic continuation of solutions from the T-region. In this approach, taking into account the mass and charge operators leads to a continuous spectra of mass and charge.

Keywords: charged black holes, cosmological constant, mass and charge function, Hamiltonian constraint, quantization, mass and charge operators.

АНОТАЦІЯ. В даній роботі розглядається задача вивчення геометрії мінісуперпростору та його зв'язок із відповідною геометрією простору-часу сферично-симетричної конфігурації електромагнітного та гравітаційного полів з урахуванням космологічної сталої та побудова хвильової функції відповідної квантової системи. З узагальненої теореми Біркгоффа випливає, що для даної системи можна визначити R- та Т-області, що спрощує опис динамічної системи. Виходячи зі стандартної класичної дії Ейнштейна-Гільберта, будується лагранжіан конфігурації полів для сферично-симетричного простору-часу. Лагранжіан системи є виродженим та містить нединамічну ступінь вільності що призводить до в'язі. Виключаючи в'язь, можна перейти до опису динамічної системи в конфігураційному просторі (мінісуперпросторі). Розглядаються додаткові фізичні величини, що збаерігаються: повна маса і заряд системи. Відзначимо, що геометрія мінісуперпростору є конформно-пласкою. Крім стандартних горизонтів, властивих зарядженим чорним дірам, простір-час має додатковий космологічний горизонт. В конфігураційному просторі найпростіші інваріанти тензора кривини такі як скалярна кривина, квадрат тензора Річі, інваріант Кречмана, дорівнюють нулю, в той час як ненульові компоненти тензора Річі і тензора кривини розбігаються на аналозі космологічного горизонта.

В рамках канонічної квантової гравітації з матеріальними джерелами, фізичні стани знаходяться шляхом розв'язання гамільтонової в'язі в операторній формі для хвильової функції системи, визначеної на мінісуперпросторі, з урахуванням додаткових величин, що зберігаються. Формальне квантування в R-області при цьому можна розглядати як аналітичне продовження розв'язків із Т-області. В обраному підході, врахування операторів маси і заряду призводить до неперервного спектру маси і заряду.

Ключові слова: заряджені чорні діри, космологічна стала, масова і зарядова функція, Гамільтонова в'язь, квантування, оператори маси і заряду.

1. Introduction

The study of the canonical formalism of general relativity shows that all dynamic information about the gravitational field is contained in constraints. After replacing momenta $P_{ij} = \delta S / \delta \gamma_{ij}$ they lead to the Einstein-Hamilton-Jacobi equations (EHJ) for action functional S:

$$G_{ijkl} \frac{\delta S}{\delta \gamma_{ij}} \frac{\delta S}{\delta \gamma_{kl}} - \sqrt{\gamma} R^{(3)} = 0 \,,$$

where $G_{ijkl} = (1/2)(\gamma)^{-1/2} \left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl} \right)$ is supermetric of configuration space and conditions of invariance $(\delta S / \delta \gamma_{ij})_{;j} = 0$.

When quantized, in accordance with the Dirac approach, constraints become conditions on the state vector (Dirac, 1979; Gitman, 1986). After replacement $\hat{P}_{ij} = -i\delta / \delta \gamma_{ij}$ momenta constraints leads to the relations of invariance of the state vector: $(\delta / \delta \gamma_{ij} \Psi)_{;j} = 0$. In this case, the Hamiltonian constraint leads to the Wheeler-DeWitt equation

$$\left\{ G_{ijkl} \frac{\delta}{\delta \gamma_{ij}} \frac{\delta}{\delta \gamma_{kl}} - \sqrt{\gamma} R^{(3)} \right\} |\Psi\rangle = 0 \,.$$

(See, for example, Schulz 2014 and references therein.)

We see that both classical and quantum aspects of the behavior of a gravitational field are determined by the metric of superspace so that superspace is an arena of action classical and quantum geometrodynamics. Studying the geometry of a general superspace, one can obtain important information about the classical and quantum manifestations of the dynamical system under consideration. However, the study of superspace in general is faced with great mathematical difficulties. Therefore, reduced models are widely used, among which spherically symmetric configurations are popular and simplest models used to study the problems of quantum gravity.

The work is devoted to the study of the minisuperspace of spherically-symmetric configurations of the electromagnetic and gravitational fields with a cosmological constant and the search for a correspondence between the space-time and minisuperspace phenomena, and their quantization. Here we consider the class of configurations with diagonal space-time metrics. We are based on the observation that the considered configurations that are stationary from the point of view of the external observer, there are certain regions of the space-time with dynamic behavior. This means that in these regions there is an evolution of the space-time geometry in time, which is responsible for the quantum mechanical properties of the considered black hole model (Nakamura, 1993; Gladush, 2016).

2. Classic description of CBH with Λ

The action for the gravitational and electromagnetic fields with cosmological constant in space-time $V^{(4)}$ has the form

$$
\begin{aligned}
S_{tot} &= -\frac{1}{16\pi} \int_{V^{(4)}} \left(\frac{c^3}{\kappa} \left(R^{(4)} + 2\Lambda \right) + \right. \quad (1) \\
&+ \left. \frac{1}{c} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g^{(4)}} d^4 x.
\end{aligned}
$$

For a spherically symmetric configuration, the electromagnetic field tensor and the interval have the forms

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \rightarrow F_{ab} = A_{b,a} - A_{a,b}, \quad (2)$$

$$ds^2 = h\left(x^0, r\right) \left(dx^0\right)^2 - g\left(x^0, r\right) dr^2 - R^2\left(x^0, r\right) d\sigma^2, \quad (3)$$

where $d\sigma^2 = d\theta^2 + \sin\theta d\alpha^2$ is the angular part of the metric; a, b = 0, 1. After integrating over the angles and discarding the surface term, the action (1) can be reduced to the form

$$
\begin{aligned}
S &= \int_{V^2} \left(\frac{c^3}{2\kappa} \sqrt{gh} \left[1 + \frac{RR_{,1}}{g} \left(\ln Rh \right)_{,1} - \Lambda R^2 - \right. \right. \quad (4) \\
&- \left. \left. \frac{RR_{,0}}{h} \left(\ln(gR) \right)_{,0} \right] + \frac{1}{2c} \frac{\left(A_{1,0} - A_{0,1} \right)^2 R^2}{\sqrt{gh}} \right) dx^2.
\end{aligned}
$$

Here $X_{,0} \equiv \partial X / \partial x^0$, $X_{,1} \equiv \partial X / \partial r$ denote the derivatives with respect to x^0 and x^1. Information on the structure of space is contained in the quantity $(\nabla R)^2 = g^{ab} R_{,a} R_{,b}$ (see Berezin, 2003). The surface $R\left(x^0, r\right) = R_g = const$ for which $(\nabla R)^2 = 0$ divides V^4 into two T- and two R-regions. Moreover, $(\nabla R)^2 > 0$ in the T-region, and $(\nabla R)^2 < 0$ in the R-region. Using the generalized Birkhoff theorem, we can choose a coordinate system in the R-region in which h, g and R depend only on the spacelike coordinate r. Similarly, in the T-region there exists a coordinate system in which h, g and R depend only on the timelike coordinate x^0. Then the metrics in the R- and T-regions take the form:

$$ds_R^2 = h\left(r\right) \left(dx^0\right)^2 - g\left(r\right) dr^2 - R^2\left(r\right) d\sigma^2, \quad (5)$$

$$ds_T^2 = h\left(x^0\right) \left(dx^0\right)^2 - g\left(x^0\right) dr^2 - R^2\left(x^0\right) d\sigma^2. \quad (6)$$

In this case, the action (4) is divided in the sum $S = S_R + S_T$, where S_R and S_T are the actions defined in the R and T regions respectively. The Lagrangians corresponding to them have the form

$$
\begin{aligned}
L_R &= \frac{\chi_T}{2c} \frac{A_{0,1}^2 R^2}{\sqrt{gh}} + \quad (7) \\
&+ \frac{\chi_T c^3}{\kappa} \sqrt{gh} \left[1 - \Lambda R^2 + \frac{RR_{,1}}{g} \left(\ln Rh \right)_{,1} \right],
\end{aligned}
$$

$$
\begin{aligned}
L_T &= \frac{\chi_R}{2c} \frac{A_{1,0}^2 R^2}{\sqrt{gh}} + \quad (8) \\
&+ \frac{\chi_R c^3}{\kappa} \sqrt{gh} \left[1 - \Lambda R^2 - \frac{RR_{,0}}{h} \left(\ln Rg \right)_{,0} \right].
\end{aligned}
$$

Here χ_T and χ_R - constants, obtained by integration of the action over coordinate x^0 in R-region and co-ordinate r in T-region. Since the Lagrangian is defined up to a constant factor, we further assume that $\chi_R = \chi_T = 1$ (meter).

It is convenient to introduce new variables in the R- and T- regions:

$$\phi_R = -A_0, \qquad \xi_R = -Rh, \qquad N_R = \sqrt{gh}\,,$$
$$\phi_T = A_1, \qquad \xi_T = Rg, \qquad N_T = \sqrt{gh}\,. \qquad (9)$$

In these variables, the Lagrangians (7) and (8) take a uniform form:

$$L = \frac{F_{,\alpha}^2 R^2}{2cN} + sN\left(1 - \Lambda R^2 + \frac{R_{,\alpha}U_\alpha}{N^2}\right). \qquad (10)$$

where $s = c^3/\kappa$, α - is the evolutionary parameter, which in each of the regions takes the form $\alpha = x^0 = ct$ or $\alpha = r$ respectively. From the Lagrange-Euler equation for variable N in (10) it follows that $\partial L/\partial N = 0$. It means that N is Lagrange multiplier and there is the constraint applied to system:

$$-\frac{\phi_{,\alpha}^2 R^2}{2cN^2} + \frac{s}{2}\left(1 - \Lambda R^2 + \frac{\xi_{,\alpha}R_{,\alpha}}{N^2}\right) = 0. \qquad (11)$$

Expressing N from (11) and substituting it in (10), we obtain new Lagrangian and action of the system:

$$L = s\sqrt{(1 - \Lambda R^2)(-R_{,\alpha}\xi_{,\alpha} + \frac{R^2}{sc}\phi_{,\alpha}^2)}, \qquad (12)$$

$$S = s\int\sqrt{(1 - \Lambda R^2)(-dRd\xi + \frac{R^2}{sc}d\phi^2)}. \qquad (13)$$

Thus, the action (13) for the system is an action for geodesic in a minisuperspace with a metric

$$d\Omega^2 = (1 - \Lambda R^2)(-dRd\xi + \frac{R^2}{sc}d\phi^2). \qquad (14)$$

It appears that minisuperspace is conformally flat, which leads to the fact, that the scalar curvature, the square of the Ricci tensor, the Kretschmann invariant, are vanish, while the components of the Ricci tensor and the curvature tensor diverge on the minisuperspace analogue of the cosmological horizon ($c\tau + x = 1/\sqrt{\Lambda}$). This can be seen by following transition to new variables:

$$\xi = \frac{1}{sc}\left(c\tau - x - \frac{y^2}{c\tau + x}\right), \qquad (15)$$
$$R = c\tau + x, \qquad \phi = \frac{y}{c\tau + x}.$$

In these variables metric (14) takes form:

$$d\Omega^2 = (1 - \Lambda(c\tau + x))(-c^2 d\tau^2 - dx^2 - dy^2). \qquad (16)$$

Introducing generalized momenta as $P_i = \frac{\partial L}{\partial Q_{,\alpha}^i}$ where L is lagrangian (10), we define Hamiltonian of the system:

$$H = P_i Q_{,\alpha}^i - L = \qquad (17)$$
$$= \frac{N}{2s}\left(-4P_R P_\xi + \frac{sc}{R^2}P_\phi - s^2(1 - \Lambda R^2)\right) = -N\frac{\partial L}{\partial N}.$$

Thus, we obtain Hamilton constraint: $H = 0$. Therefore, for further consideration, it's expediently to introduce additional physical quantities - the mass and charge of the system. In the classical case, the charge is determined by the charge function as follows (Gladush, 2017):

$$Q = \frac{R^2}{\sqrt{gh}}(A_{0,1} - A_{1,0}). \qquad (18)$$

The total mass of the configuration has the form (Gladush, 2012):

$$M = \frac{s}{c}R\left(1 - \frac{\Lambda}{3}R^2 + \frac{R_{,0}^2}{h} - \frac{R_{,1}^2}{g}\right) + \frac{Q^2}{2c^2R}. \qquad (19)$$

In new variables these functions have following form:

$$Q = cP_\phi, \quad M = \frac{s}{2c}\left(R - \frac{\Lambda}{3}R^3 + \frac{4\xi P_\xi^2}{s^2}\right) + \frac{P_\phi^2}{2R}. \quad (20)$$

In order to obtain classical solution, we substitute following relations in Hamilton constraint (17) and mass and charge functions:

$$P_\xi = \frac{\partial S}{\partial \xi}, \quad P_R = \frac{\partial S}{\partial R}, \quad P_\phi = \frac{\partial S}{\partial \phi}, \qquad (21)$$

This leads to the Einstein-Hamilton-Jacobi equation and equations for total mass and charge of the system:

$$-4\frac{\partial S}{\partial \xi}\frac{\partial S}{\partial R} + \frac{sc}{R^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + s^2(1 - \Lambda R^2) = 0, \quad (22)$$

$$c\frac{\partial S}{\partial \phi} = q, \quad \frac{s}{2c}\left(R - \frac{\Lambda}{3}R^3 + \frac{4\xi}{s^2}\left(\frac{\partial S}{\partial \xi}\right)^2\right) = m. \quad (23)$$

This equations have following solution:

$$S = \frac{q}{c}\phi + p_\xi \xi - \frac{s^2}{4p_\xi}\left(R - \frac{\Lambda}{3}R^3 + \frac{q^2}{2cR}\right). \qquad (24)$$

Here it is taken into account that from the canonical equations for momenta P_ξ and P_ϕ it follows, that they are integrals of motion. In particular, $cP_\phi = q$ - charge of the system. Calculating derivatives $\partial S/\partial q$ and $\partial S/\partial p_\xi$ and equating them to constant, we find trajectories:

$$\xi = \xi_0 - \frac{s^2}{4p_\xi^2}\left(R - \frac{\Lambda}{3}R^3 + \frac{q^2}{2cR}\right), \quad \phi = \frac{sq}{2cp_\xi R}. \quad (25)$$

From mass function we find out that $p_\xi = \sqrt{\frac{msc}{2\xi_0}}$. Finally, we get:

$$\xi = \xi_0\left(1 - \frac{s}{2mc}\left(R - \frac{\Lambda}{3}R^3 + \frac{q^2}{scR}\right)\right), \quad \phi = \sqrt{\frac{s\xi_0}{2mc^3}}\frac{q}{R}.$$
$$(26)$$

Performing reverse transformation to (9), we obtain following metric coefficients:

$$h_R = \frac{2mc\xi_0}{s}\left(1 - \frac{2mc}{sR} + \frac{q^2}{scR^2} - \frac{\Lambda}{3}R^3\right), \quad (27)$$

$$g_R = \frac{N_R^2}{h_R}; \qquad h_T = \frac{N_T^2}{g_T},$$

$$g_T = -\frac{2mc\xi_0}{s}\left(1 - \frac{2mc}{sR} + \frac{q^2}{scR^2} - \frac{\Lambda}{3}R^3\right).$$

Assuming $N_R = N_T = 1, \xi_0 = s/2mc$, we obtain classical result for Reissnerвъ"Nordstromвъ"de Sitter metrics. Solution (26) can be written in terms of minisuperspace variables $(c\tau, x, y)$ in parametric form:

$$c\tau = \tfrac{1}{2}\left(\left(1 - \tfrac{a}{R_g}\right)R + a + \tfrac{a}{R_g}\tfrac{\Lambda}{3}R^3\right), \qquad (28)$$

$$x = \tfrac{1}{2}\left(\left(1 + \tfrac{a}{R_g}\right)R - a - \tfrac{a}{R_g}\tfrac{\Lambda}{3}R^3\right),$$

$$y = \sqrt{\tfrac{\xi_0}{R_g}}q.$$

Here $a = sc\xi_0$, $R_g = 2mc/s$. The curves that are described by (28) are third-order curves. Their intersection with the light cone in the minisuperspace correspond to the event horizons in space-time, while the interior of the cone corresponds to the T-regions. Further, after second intersection with a conus, they move outside the light cone of the miniscuperspace, which corresponds to the spatial evolution of fields in the outer R-region. Then they are returning again to the T-region. For extremal and superextremal charged black holes, the corresponding curves are tangent to or pass by the cone, respectively. All this points us to the connection between the geometry of the minisuperspace and the geometry of spherically-symmetric space-time configurations.

3. Quantum description of CBH with Λ

In order to build quantum description of the system, we proceed form functions of physical quantities to their operators. Defining coordinates and momenta operators as

$$\hat{q}^i = q^i, \qquad \hat{p}_i = -i\hbar\frac{\partial}{\partial q^i}, \qquad (29)$$

we obtain following Hamilton, mass and charge operators:

$$\hat{H} = -\frac{N}{2s}\left(4\hbar^2\frac{\partial^2}{\partial\xi\partial R} + \hbar^2\frac{sc}{R^2}\frac{\partial^2}{\partial\phi^2} + s^2(1 - \Lambda R^2)\right), (30)$$

$$\hat{M} = \frac{s}{c}\left(R - \frac{\Lambda}{3}R^3 - 4\frac{\hbar^2}{s^2}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}\right) - \frac{\hbar^2}{2R}\frac{\partial^2}{\partial\phi^2},$$

$$\hat{Q} = -i\hbar c\frac{\partial}{\partial\phi}.$$

In order for the total mass operator to be Hermitian, we use the following ordering of the operators: $p_\xi\xi p_\xi$. Obtained operators have following commutators:

$$\left[\hat{M}, \hat{Q}\right] = 0, \quad \left[\hat{H}, \hat{Q}\right] = 0, \quad \left[\hat{H}, \hat{M}\right] = \frac{\hbar^2}{sc}\frac{\partial}{\partial\xi}\hat{H} \sim 0. \qquad (31)$$

By (31) we can require that the wave function of the system satisfy following system of equations:

$$\begin{cases} \hat{H}\Psi = 0, \\ \hat{M}\Psi = m\Psi, \\ \hat{Q}\Psi = q\Psi. \end{cases} \qquad (32)$$

Jointly solving (32), we obtain wave function of the system, which is regular on horizons:

$$\Psi(\xi, R, \phi) = CJ_0\left(\sqrt{\xi R\left[-1 + \frac{2mc}{sR} - \frac{q^2}{scR^2} + \frac{\Lambda}{3}R^3\right]}\right). \qquad (33)$$

Since system (32) has a solution for any values of m and q, the mass and the charge spectra of black hole are continuous.

4. Conclusion

Obtained result is consistent with the previous result of one of the authors (Gladush, 2016; Gladush, 2017) and turns into it at $\Lambda = 0$. Considered approach leads to continuous spectra of mass and charge, what is coincides with results of other authors (Kuchar, 1994; Louko, 1996; Nakamura, 1993). Also quantization was carried out in uniform way, for R- and T-regions simultaneously. Since quantization has physical sense only in T-regions, the results obtained for the R-regions can be considered an analytical continuation of the T-regions solutions. Without the imposition of additional differential-geometric and group structures on the space-time, configuration or phase spaces, it is impossible to obtain a discrete spectrum. The reason for this is that the above differential equations determine only the local structure of the space, whereas the global structure needs to be defined. Thus, the question of the mass and charge spectra is not solved by a local approach, since the properties of the spectra depend on the global properties of the space-time and minisuperspace geometry, and the structure of the phase space of the considered system.

References

Dirac P.: 1979, *Lectures on quantum mechanics*, in Principles of quantum mechanics, (Nauka, Moscow).

Gitman D., Tyutin I.: 1986, *Canonical quantization of fields with constraints*, (Nauka, Moscow), 216.

Schulz B.: 2014, arXiv:1409.7977, 100.

Berezin B.A.: 2003, *Fizika Elementarnykh Chastits i Atomnogo Yadra* **34**, 49.

Nakamura K., Konno S., Oshiro Y., Tomimatsu A.: 1993, *Prog. Theor. Phys.* **90**, 861.

Gladush V.: 2016, *Visn. Dnipr. univ.*, **24**, 31.

Kuchař K.: 1994, *Phys. Rev.*, **D50**, 3961.

Louko J., Winters-Hilt S.: 1996, *Phys. Rev.*, **D54**, 2647.

Gladush V.D.: 2012, *STFI*, **1**, 48.

Gladush V.D.: 2017, *Odessa Astr. Publ.*, **30**, 19.