QUANTUM MODEL OF A CHARGED BLACK HOLE

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A canonical approach for constructing of the classical and quantum description spherically-symmetric configuration gravitational and electromagnetic fields is considered. According to the sign of the square of the Kodama vector, space-time is divided into R- and T-regions. By virtue of the generalized Birkhoff theorem, one can choose coordinate systems such that the desired metric functions in the T-region depend on the time, and in the R-domain on the space coordinate. Then, the initial action for the configuration breaks up into terms describing the fields in the T- and R-regions with the time and space evolutionary variable, respectively. For these regions, Lagrangians of the configuration are constructed, which contain dynamic and non-dynamic degrees of freedom, leading to constrains. We concentrate our attention on dynamic T-regions. There are two additional conserved physical quantities: the charge and the total mass of the system. The Poisson bracket of the total mass with the Hamiltonian function vanishes in the weak sense. A classical solution of the field equations in the configuration space (minisuperspace) is constructed without fixing non-dynamic variable. In the framework of the canonical approach to the quantum mechanics of the system under consideration, physical states are found by solving the Hamiltonian constraint in the operator form (the DeWitt equation) for the system wave function Ψ. It also requires that Ψ is an eigenfunction of the operators of charge and total mass. For the symmetric of the mass operator the corresponding ordering of operators is carried out. Since the total mass operator commutes with the Hamiltonian in the weak sense, its eigenfunctions must be constructed in conjunction with the solution of the DeWitt equation. The consistency condition leads to the ansatz, with the help of which the solution of the DeWitt equation for the state Ψ_{em} with a defined total mass and charge is constructed, taking into account the regularity condition on the horizon. The mass and charge spectra of the configuration in this approach turn out to be continuous. It is interesting that formal quantization in the R-region with a space evolutionary coordinate leads to a similar result.

Keywords: charged black holes, mass and charge functions, constraints, mass and charge operators, the compatibility condition.

1. Introduction

Spherically-symmetric systems of gravitational and electromagnetic fields represent the simplest configurations for check of the main ideas and results of quantum gravitation. They are a convenient testing ground for studying some of the problems arising from a rigorous consideration of a more complete theory. One of the main features general-relativistic configurations is their degeneracy. The general formalism of the canonical approach to degenerate systems was constructed by Dirac [1], which was then developed in many works (see, for example, [2,3,4]). Problems of the quantum description spherically-symmetric configurations gravitational and electromagnetic fields was considered in many work A general, geometrodynamical approach to the spherically symmetric gravitational field was developed in [5], which was then generalized to the case of a spherically symmetric configuration electromagnetic and gravitational fields in [6].

The proposed model is based on the observation that the classical spherically-symmetric configurations of electromagnetic and gravitational fields are stationary from the point of view of an external observer, have certain regions of space-time with dynamic behavior. This means that these regions do not allow a timelike vector Kodama (Killing) [7], which implies in these regions the evolution of the geometry of space-time in time. This evolution of a space-time geometry is also responsible for quantum-mechanical properties of the considered model of the charged black hole. Such models, with a fixed evolutionary time coordinate and space-like Killing vector were considered in [8, 9]. In this paper, on the basis of simple approach using the DeWitt equation and quantum mass and charge operators, a quantum description of the spherically symmetric configuration of the gravitational and electromagnetic fields is constructed, i.e. quantum model of a charged black hole with a continuous spectrum of masses and charge.

2. Classical description of the spherically-symmetric configuration of the gravitational and electromagnetic fields

Consider the spherically symmetric space-time $M^{(4)}$ with the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ab}dx^a dx^b - R^2 d\sigma^2 , \quad (1)$$

$$d\sigma^2 = d\theta^2 + \sin^2\theta d\alpha^2 . \quad (2)$$

Here $R = R(x^a)$, $\gamma_{ab} = \gamma_{ab}(x^a)$ — 2D metric tensor, $\sqrt{-g} = \sqrt{-\gamma}R^2\sin\theta$, где $g = \det|g_{\mu\nu}|$, $\gamma = \det|\gamma_{ab}|$, $\mu,\nu = 0,1,2,3$; $a,b = 0,1$.

The total action for a system of gravitational and electromagnetic fields has the form

$$S = -\frac{1}{16\pi c}\int_{M^{(4)}}\left(\frac{c^4}{\kappa}\,{}^{(4)}R + F^{\mu\nu}F_{\mu\nu}\right)\sqrt{-g}d^4x . \quad (3)$$

Here $\,{}^{(4)}R$ is the scalar curvature of $M^{(4)}$ with respect to the metric $g_{\mu\nu}$, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor, where $A_\mu = \{A_a, 0, 0\}$ is vector potential.

In the spherically symmetric case, after integrating over the angles and discarding the surface term, the action (3) can be represented in the form

$$S = -\frac{1}{4c}\int_{M^{(2)}}\left\{\frac{c^4}{\kappa}\sqrt{-\gamma}\left(R^2\,{}^{(2)}R + 2(\nabla R)^2 - 2\right)\right.$$
$$\left. - \frac{2}{\sqrt{-\gamma}}R^2 E^2\right\}d^2x , \quad (4)$$

where $\,{}^{(2)}R$ is scalar curvature $M^{(2)}$ (radially-time part $M^{(4)}$), $(\nabla R)^2 = \gamma^{ab}R_{,a}R_{,b}$. The reduced action is invariant under coordinate transformations $x^a = x^a(\acute{x}^b)$.

In a spherically-symmetric space-time there is a preferable reference system (RS). Tangents to world lines of this RS are proportional to Kodama's vector

$$\vec{K} = K^a\partial_a = -e^{ab}R_{,b}\partial_a . \quad (5)$$

It is easy to see that $\vec{K}R = K^a R_{,a} = 0$ and the Kodama vector satisfies the continuity equation $K^a_{;a} = 0$. For the free space, as well as for electrovacuum spaces, the Kodama vector is transformed into a Killing vector, which corresponds to the generalized Birkhoff theorem (Frolov, 1998).

Using an admissible coordinate transformation, we lead the metric (1) to the diagonal form

$$ds^2 = f(r,x^0)(dx^0)^2 - h(r,x^0)dr^2 - R^2(r,x^0)d\sigma^2 , \quad (6)$$

where $f > 0, h > 0$. Then $\sqrt{-\gamma} = \sqrt{fh}$ and the action takes the form

$$S = \frac{1}{2c}\int_{M^{(2)}}\left\{\frac{c^4}{\kappa}\sqrt{hf}\left\{\frac{R}{h}R_{,1}\left(\ln(fR)\right)_{,1} - \right.\right.$$
$$\left.\left. - \frac{R}{f}R_{,0}\left(\ln(hR)\right)_{,0} + 1\right\} + \frac{R^2}{\sqrt{fh}}E^2\right\}d^2x , \quad (7)$$

where $R_{,0} = \partial R/\partial x^0$, $R_{,1} = \partial R/\partial r$. In what follows we confine ourselves to the class of diagonal metrics.

Note that information about the structure of $M^{(2)}$ is contained in the square of the Kodama vector

$$(\vec{K})^2 = -(\nabla R)^2 = \frac{1}{h}\left(R_{,1}\right)^2 + \frac{1}{f}\left(R_{,0}\right)^2 . \quad (8)$$

Light surfaces $R(r,x^0) = R_g = const$, for which $(\vec{K})^2 = -(\nabla R)^2 = 0$, divide $M^{(2)}$ into regions

R-regions $M_R^{(2)} \subset M^{(2)}$, when $(\vec{K})^2 > 0$,

T-regions $M_T^{(2)} \subset M^{(2)}$, when $(\vec{K})^2 < 0$.

In the R-region the surface $R(r,x^0) = const$ are timelike, and in the T-region is spacelike. In the T-region we sometimes use the notation $R = cT$. Using the generalized Birkhoff theorem, in the R-region we can choose a coordinate system (CS) in which γ_{ab} and R depend only on the space-like coordinate r. Similarly, in the T-region, there exists an CS in which γ_{ab} and R depend on the time-like coordinate x^0.

In the T-region it is convenient to go to the new variable

$$f(x^0) = \frac{N^2(x^0)}{h(x^0)} , \qquad A_1 = \phi , \quad (9)$$

so that the metric (6) takes the form

$$ds_T^2 = \frac{N^2(x^0)}{h(x^0)}(dx^0)^2 - h(x^0)dr^2 - R^2(x^0)d\sigma^2 . \quad (10)$$

In this case for the action (7) we get

$$S_T = \int_{x^0 \in M_T^{(2)}} L_T dx^0 , \quad (11)$$

$$L_T = \frac{l}{2c}\left\{\frac{1}{N}\left[R^2\left(\phi_{,0}\right)^2 - \right.\right.$$
$$\left.\left. - \frac{c^4}{\kappa}\left(RR_{,0}h_{,0} + h\left(R_{,0}\right)^2\right)\right] + \frac{c^4}{\kappa}N\right\} , \quad (12)$$

where L_T is the Lagrangian of the reduced system. Here, $l = r_2 - r_1$ is a constant that arises as a result of integration over r in the range from r_1 to r_2, permissible transformations are $x^0 = x^0(\acute{x}^0)$.

Further, it is convenient to go over to the so-called characteristic variables $\{\xi = hR, \ R = cT, \ \phi\}$. Then the metric and Lagrange function of the T-region take the form

$$ds^2 = N^2\frac{R}{\xi}\left(dx^0\right)^2 - \frac{\xi}{R}dr^2 - R^2 d\sigma^2 , \quad (13)$$

$$L = \frac{l}{2c}\left\{\frac{1}{N}\left[-\frac{c^4}{\kappa}\dot{\xi}\dot{R} + R^2\dot{\phi}^2\right] + \frac{c^4}{\kappa}N\right\} . \quad (14)$$

In the case of the R-region, we pass to the variables

$$h(r) = \frac{N_R^2(r)}{f(r)} , \qquad \{\eta = fR, A_0 = \varphi\} , \quad (15)$$

in which the metric and Lagrange functions in the characteristic variables take the form

$$L_R = \frac{l}{2c} \left\{ \frac{1}{N_R} \left[\frac{c^4}{\kappa} \dot{R}\acute{\eta} + R^2 \dot{\varphi} \right]^2 + \frac{c^4}{\kappa} N_R \right\}, \quad (16)$$

$$ds_R^2 = \frac{\eta}{R}(dx^0)^2 - N_R^2 \frac{R}{\eta} dr^2 - R^2 d\sigma^2. \quad (17)$$

From the Lagrangian (14) follows the primary constraint $P_N = \partial L/\partial \dot{N} = 0$ and momenta $P_i = \partial L/\partial \dot{q}^i$:

$$P_\xi = -\frac{c^3 l}{2\kappa N} \dot{R}, \quad P_R = -\frac{c^3 l}{2\kappa N} \dot{\xi}, \quad P_\phi = \frac{l}{cN} R^2 \dot{\phi}. \quad (18)$$

The Hamiltonian function $H = P_\xi \dot{\xi} + P_R \dot{R} + P_\phi \dot{\phi} - L$ leads to the secondary constraint in the T-region

$$H = \frac{Nc}{2l} \left\{ -\frac{4\kappa}{c^4} P_\xi P_R + \frac{1}{R^2} P_\phi^2 - \frac{l^2 c^2}{\kappa} \right\} \sim 0 \quad (19)$$

in the characteristic variables.

The Maxwell equations following from actions (3) or (4) lead to to the relation $(R^2/NE)_{,b} = 0$. This implies the conservation law

$$\frac{R^2}{N} E = Q = const. \quad (20)$$

It is natural to define the following function:

$$Q(N, R, \dot{\phi}) = \frac{R^2}{N} \dot{\phi} = \frac{c}{l} P_\phi. \quad (21)$$

This function for a free electromagnetic field is conserved and is equal to the configuration charge inside the region of radius R. In what follows we will call it the charge function.

We introduce the mass function (Cahill,1970; Berezin, 1987; Gladush, 2012) by the relation:

$$M_f(\gamma_{ab}, R) = \frac{c^2}{2\kappa} R \left(1 + \gamma^{ab} R_{,a} R_{,b} \right). \quad (22)$$

In the R-region, it is related to the field energy of a spherical region of radius R. Its value for the free gravitational field is constant: $M_f = m$ and determines the mass that appears in the Newtonian limit of the gravitational field. In the T-region, it is conserved and in the used variables has the form

$$M_f = \frac{c^2}{2\kappa} \left(R + \frac{1}{N^2} \xi \dot{R}^2 \right). \quad (23)$$

For the system under consideration, $M_f \neq const$. It can be shown that the quantity given by formula

$$M_{tot} = M_f + \frac{Q^2}{2c^2 R} = const, \quad (24)$$

is conserved and has the meaning of the total field mass of the spherical region of radius R taking into account a contribution of the electromagnetic field. In characteristic variables, as well as through momenta, it has the form

$$M_{tot} = \frac{c^2}{2\kappa} \left[R + \frac{1}{N^2} \left(\xi \dot{R}^2 + \frac{\kappa R^3 \dot{\phi}^2}{c^4} \right) \right], \quad (25)$$

$$M_{tot} = \frac{1}{2l^2} \left[\frac{l^2 c^2}{\kappa} R + \frac{4\kappa}{c^4} \xi P_\xi^2 + \frac{1}{R} P_\phi^2 \right]. \quad (26)$$

We write out the Poisson brackets of the dynamic quantities:

$$\{H, M_{tot}\} = \frac{2\kappa}{l^2 c^4} P_\xi H \sim 0, \quad \{H, Q\} = \{M_{tot}, Q\} = 0.$$

The Lagrangian multiplier N can be excluded from the action (11), (12). Then the initial variational principle is transformed into the variational principle in the configuration space (Gowdy, 1970). We rewrite the Lagrange function (14) in the form

$$L_T = \frac{l}{2c} \left\{ \frac{\mathfrak{T}}{N} + NU \right\}, \quad (27)$$

where

$$\mathfrak{T} = -\frac{c^4}{\kappa} \dot{\xi} \dot{R} + R^2 \dot{\phi}^2, \qquad U = \frac{c^4}{\kappa}. \quad (28)$$

Then we have

$$\frac{\partial L_T}{\partial N_T} = \frac{l}{2c} \left\{ -\frac{\mathfrak{T}}{N_T^2} + U \right\} = 0 \quad (29)$$

This implies $N_T = \sqrt{\mathfrak{T}/U}$ and from (11) we obtain action for a geodesic in a minisuperspace

$$S = \frac{lc}{\sqrt{\kappa}} \int \sqrt{-\frac{c^4}{\kappa} d\xi dR + R^2 (d\phi)^2} = \frac{lc}{\sqrt{\kappa}} \int d\Omega \quad (30)$$

with the metric

$$d\Omega^2 = -\frac{c^4}{\kappa} d\xi dR + R^2 (d\phi)^2. \quad (31)$$

The geodesic equations obtained from here, together with the Hamiltonian constraint, are equivalent to the original system of Einstein's equations. It turns out that the corresponding configuration space with the metric (31 is flat, since the curvature tensor for the metric (31) vanishes. Let us write out the volume element defining the measure in the configuration space:

$$dV = \sqrt{-\det \|\Omega_{AB}\|} dq^1 dq^2 dq^3 = \frac{c^2}{2\kappa} R d\xi dR d\phi. \quad (32)$$

3. Quantum description of the spherically-symmetric configuration of the gravitational and electromagnetic fields

The quantum states of the field configuration under consideration are determined by the wave function $\Psi(R, \xi, \phi)$ on the minisuperspace with the coordinates $\{R, xi, phi\}$. The corresponding momentum operators in this representation have the form:

$$\hat{P}_R = -i\hbar \frac{\partial}{\partial R}, \quad \hat{P}_\xi = -i\hbar \frac{\partial}{\partial \xi}, \quad \hat{P}_\phi = -i\hbar \frac{\partial}{\partial \phi}. \quad (33)$$

The classical Hamiltonian, the total mass and charge functions lead to operators

$$\hat{H} = \frac{Nc}{2l} \left\{ \frac{4\kappa\hbar^2}{c^4} \frac{\partial^2}{\partial R \partial \xi} - \frac{\hbar^2}{R^2} \frac{\partial^2}{\partial \phi^2} - \frac{c^2 l^2}{\kappa} \right\}, \quad (34)$$

$$\hat{M} = \frac{1}{2l^2} \left(\frac{l^2 c^2}{\kappa} R - \frac{4\kappa\hbar^2}{c^4} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} - \frac{\hbar^2}{R} \frac{\partial^2}{\partial \phi^2} \right). \quad (35)$$

$$\hat{Q} = \frac{c}{l} \hat{P}_\phi = -i \frac{c\hbar}{l} \frac{\partial}{\partial \phi}. \quad (36)$$

For the Hermitian operator of the total mass, in the configuration space with measure (32) we use the following ordering of the operators: $\hat{P}_\xi \xi \hat{P}_\xi$. The following commutation relations hold

$$\left[\hat{H}, \hat{M} \right] = -\frac{2\kappa\hbar^2}{l^2 c^4} \frac{\partial}{\partial \xi} \hat{H} \sim 0, \quad \left[\hat{H}, \hat{Q} \right] = \left[\hat{Q}, \hat{M} \right] = 0.$$

States with a certain total mass and charge correspond to eigenfunctions and eigenvalues of the operators of total mass and charge:

$$\hat{M}\Psi_m = m\Psi_m, \qquad \hat{Q}\Psi_q = q\Psi_q. \quad (37)$$

They reduce to the following equations

$$\left\{ \frac{c^2 l^2}{\kappa} R - \frac{4\kappa\hbar^2}{c^4} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} - \frac{\hbar^2}{R} \frac{\partial^2}{\partial \phi^2} \right\} \Psi_m = 2l^2 m\Psi_m. \quad (38)$$

$$\frac{\partial}{\partial \phi} \Psi_q = \frac{iql}{c\hbar} \Psi_q. \quad (39)$$

From the last equation we obtain $\Psi_q = A e^{i(ql/c\hbar)\phi}$. Now, the general wave functions of the DeWitt equation $\hat{H}\Psi = 0$ and the charge operator, also as general wave functions of the operators total mass and charge, can be represented in the form

$$\Psi = \psi(\xi, R) e^{i(ql/c\hbar)\phi}, \quad \Psi_m = \psi_m(\xi, R) e^{i(ql/c\hbar)\phi}.$$

The functions ψ and ψ_m satisfy the equations

$$\left(\frac{4\partial^2}{\partial R \partial \xi} + \frac{c^2 q^2 l^2}{\kappa\hbar^2} \frac{1}{R^2} \right) \psi = \frac{c^6 l^2}{\kappa^2 \hbar^2} \psi, \quad (40)$$

$$\left\{ \frac{l^2 c^2}{\kappa} R - \frac{4\kappa\hbar^2}{c^4} \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} + \frac{1}{R} \frac{q^2 l^2}{c^2} \right\} \psi_m = 2ml^2 \psi_m. \quad (41)$$

Next, we introduce Planckian and dimensionless quantities

$$m_{pl}^2 = c\hbar/\kappa, \ l_{pl}^2 = \hbar\kappa/c^3, \ q_{pl} = m_{pl}\sqrt{\kappa} = \sqrt{c\hbar},$$

$\mu = m/m_{pl}, \ \sigma = q/q_{pl}, \ x = \xi/l_{pl}, \ y = R/l_{pl}, \ \chi = l/l_{pl}$. Then, the system of equations (40), (41) can be rewritten as follows

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\chi^2}{4} \left(1 - \frac{\sigma^2}{y^2} \right) \psi, \quad (42)$$

$$\frac{\partial^2 \psi_m}{\partial x^2} = -\frac{1}{x} \frac{\partial \psi_m}{\partial x} + \frac{\chi^2}{4x} \left(y + \frac{\sigma^2}{y} - 2\mu \right) \psi_m. \quad (43)$$

A joint solution of this system, which regularly on the horizon, gives the wave function of configuration for the T-region in the state with a given mass m and a charge q

$$\Psi_{m,q} = C J_0 \left(\frac{lc}{l_{pl}^2} T \sqrt{h F_T(T, m, q)} \right) e^{i(ql/c\hbar)\phi}, \quad (44)$$

where $R = cT$ and

$$h = \frac{\xi}{R} = \frac{c^2}{2\kappa m} \xi_0 F_T(T, m, q) > 0, \quad (45)$$

$$F_T(T, m, q) = -1 + \frac{2\kappa m}{c^3 T} - \frac{\kappa q^2}{c^6 T^2} > 0. \quad (46)$$

The functions h and $T = R/c$ in the initial metric (10) are arbitrary here. We note that the coefficient N does not enter the wave function $\Psi_{m,q}(h, T, \phi)$, which determines the probability amplitude of the given configuration $\{h, T, \phi; m, q\}$, that is, points $\{h, T, \phi\}$ in the configuration space for the given observables m, q. The mass and charge spectra in this approach are continuous. We note that formal quantization in the R-region with a spacelike evolutionary coordinate gives an analogous wave function.

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