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# THE ABC OF COSMOGRAPHY

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**ABSTRACT.** The basis of this work is the scheme for describing the universe, called the "cosmography" entirely based on the cosmological principle. Within the framework of such a scheme the parameters of any model that satisfies the cosmological principle (the universe is homogeneous and isotropic on large scale), can be expressed through cosmographic parameters. We demonstrate a number of advantages of the approach used before traditional methods

**Keywords:** cosmographic parameters, cosmography.

## 1. Introduction

The fundamental characteristics used to describe the evolution of the universe can be either kinematic if they are extracted directly from the space-time metric or dynamical if they depend on the properties of the fields that fill the universe. The dynamic characteristics are, of course, model dependent while the kinematic characteristics are more universal. In addition, the latter are free from the uncertainties arising when physical quantities such as, for example, energy densities are measured. It is for this reason that the kinematic characteristics are convenient for describing the current expansion of the universe. The kinematics of cosmological expansion of a homogeneous and isotropic universe has been called cosmography

In the early 60s of the last century, Alan Sandage (Sandage, 1962) defined as the primary goal of the cosmologists a search for two parameters, namely, the Hubble parameter and the deceleration parameter. However, an expansion with a constant acceleration is not the only possible realization of the kinematics of a nonstationary universe. As the universe evolves, the relative content of the components that fill it is changing and, as a consequence, the dynamics of expansion changes, hence the changes in acceleration. Thus, for a more complete description of the kinematics of cosmological expansion, it is useful to consider an extended

set of parameters by including temporal derivatives of a higher-order scale factor (Visser, 2005; Dunsby et al., 2016; Bolotin et al., 2012):

$$\begin{aligned}
 lH(t) &\equiv \frac{1}{a} \frac{da}{dt}; \\
 q(t) &\equiv -\frac{1}{a} \frac{d^2a}{dt^2} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-2}; \\
 j(t) &\equiv \frac{1}{a} \frac{d^3a}{dt^3} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-3}; \\
 s(t) &\equiv \frac{1}{a} \frac{d^4a}{dt^4} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-4}; \\
 l(t) &\equiv \frac{1}{a} \frac{d^5a}{dt^5} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-5}.
 \end{aligned} \tag{1}$$

The inclusion of higher derivatives from the scale factor, on one hand, reflects the continuous progress of observational cosmology, and, on the other, it is dictated by the need to describe the increasingly complex effects used for obtaining precise information.

## 2. The basic cosmographic relations

Let us give a number of useful relationships needed to describe the kinematics of cosmological expansion. The deceleration parameter is related to the Hubble parameter by the following relations:

$$\begin{aligned}
 lq(t) &= \frac{d}{dt} \left( \frac{1}{H} \right) - 1; \\
 q(z) &= \frac{1+z}{H} \frac{dH}{dz} - 1; \\
 q(z) &= \frac{1}{2} \frac{d \ln H^2}{d \ln(1+z)} - 1.
 \end{aligned} \tag{2}$$

Derivatives  $\frac{dH}{dz}$ ,  $\frac{d^2H}{dz^2}$ ,  $\frac{d^3H}{dz^3}$ ,  $\frac{d^4H}{dz^4}$  and can be expressed through the deceleration parameter and other cosmo-

logical parameters as follows:

$$\begin{aligned}\frac{dH}{dz} &= \frac{1+q}{1+z}H; \\ \frac{d^2H}{dz^2} &= \frac{j-q^2}{(1+z)^2}H; \\ \frac{d^3H}{dz^3} &= \frac{H}{(1+z)^3}(3q^2+3q^3-4qj-3j-s); \\ \frac{d^4H}{dz^4} &= \frac{H}{(1+z)^4}(-12q^2-24q^3-15q^4+32qj+ \\ &+ 25q^2j+7qs+12j-4j^2+8s+l). \quad (3)\end{aligned}$$

Let  $C_n \equiv \gamma_n \frac{a^{(n)}}{aH^n}$ , where  $a^{(n)} \equiv \frac{d^n a}{dt^n}$  is the  $n$ -th time derivative of the scale factor,  $n \geq 2$  and  $\gamma_2 = -1$ ,  $\gamma_n = 1$  for  $n > 2$ . Then  $C_2 = q$ ,  $C_3 = j$ ,  $C_4 = s \dots$ . For the derivatives of the parameters with respect to the redshift, the following relation takes place:

$$(1+z) \frac{dC_n}{dz} = -\frac{\gamma_n}{\gamma_{n+1}} C_{n+1} + C_n - nC_n(1+q). \quad (4)$$

Using  $dz/dt = -(1+z)H$ , the redshift derivatives can be converted into time derivatives.

### 3. Cosmography of cardassian model

Dunajski and Gibbons (Dunajski and Gibbons, 2008) proposed an original approach for testing cosmological models that satisfy the cosmological principle. Implementation of the method implies the following sequence of steps:

1. The first Friedman equation is transformed to the ODE for the scale factor. This is achieved by using the conservation equation for each component included in the model to find the dependence of the energy density on the scale factor.

2. The resulting equation is differentiated (with respect to cosmological time) as many times as the number of free parameters of the model.

3. The time derivatives of the scale factor are expressed through the cosmographic parameters.

4. By solving the obtained system of linear algebraic equations, we express all free parameters of the model through cosmographic parameters.

The procedure under consideration can be made more universal and effective if the system of Friedmann equations for the Hubble parameter  $H$  and its time derivative  $\dot{H}$  is considered as a starting one. By differentiating the equation the required number of times (this number is determined by the number of free parameters of the model), we obtain a system of equations including higher time derivatives of the Hubble parameter  $\ddot{H}, \dddot{H}, \dots$ . These derivatives are directly related to the cosmological parameters by the relations (3). We implement this procedure for the so-called Cardassian model, whose evolution is described by a system

of equations (Freese and Lewis, 2002)

$$H^2 = A\rho + B\rho^n. \quad (5)$$

$$\dot{\rho} + 3H\rho = 0. \quad (6)$$

Here  $\rho$  is the density of nonrelativistic matter. Differentiating equation (5) with respect to the cosmological time and using (6), we construct a system of coupled equations

$$\begin{aligned}H^2 &= A\rho_m + B\rho_m^n, \\ -\frac{2}{3}\dot{H} &= A\rho_m + Bn\rho_m^n, \\ \frac{2}{9}\frac{\dot{H}}{H} &= A\rho_m + n^2B\rho_m^n.\end{aligned} \quad (7)$$

Using the solutions of this system and the time derivatives of the Hubble parameter (3), we find for constants  $B$  and  $n$ :

$$\begin{aligned}\frac{B\rho_0^n}{H_0^2} &= \frac{1}{3}(1-2q_0) \\ n &= \frac{2}{3}\frac{j_0-1}{2q_0-1}.\end{aligned} \quad (8)$$

These relationships solve the problem of finding cardassian model parameters. A similar procedure can be applied to any model that satisfies the cosmological principle

Otherwise, we must treat the time-dependent solution for the density  $\rho_m$ . It can be represented in the form

$$\frac{\rho_m}{\rho_c} = \frac{-n + \frac{2}{3}(1+q)}{1-n}, \quad \rho_c \equiv \frac{3H^2}{8\pi G}. \quad (9)$$

The current density  $\rho_{m0}$  can be found by substitution  $q \rightarrow q_0$ ,  $H \rightarrow H_0$ .

It is interesting to note that the expression (7) for the parameter  $n$  coincides exactly with the parameter  $s$ , one of the so-called statefinder parameters  $\{r, s\}$  (Sahni V. et al., 2003).

$$r \equiv \frac{\dot{a}}{aH^3}, \quad s = \frac{2}{3}\frac{r-1}{2q-1}. \quad (10)$$

The coincidence is obvious, since  $r \equiv j$ . The reason for the coincidence can be explained as follows. In any model with the scale factor  $a \propto t^\alpha$ , there are the simple relations for the cosmographic parameters  $q$  and  $j$ ,

$$2q-1 = \frac{2-3\alpha}{\alpha}, \quad j-1 = \frac{2-3\alpha}{\alpha^2}. \quad (11)$$

In cardassian model  $a \propto t^{\frac{2}{3n}}$ , from which it follows that  $s = n$ .

### 4. Summary: advantages of cosmographic description

The proposed approach to finding the parameters of cosmological models has many advantages. Let's briefly dwell on them.

1. Universality: the method is applicable to any model that satisfies the cosmological principle. The procedure can be generalized to the case of models

with interaction between components (Bolotin et al., 2016).

2. Reliability: all the obtained results are accurate, since they follow from identical transformations.

3. The simplicity of the procedure.

4. Parameters of different models are expressed through a universal set of cosmological parameters. There is no need to introduce additional parameters.

5. The method provides an interesting possibility of calculating the highest cosmological parameters from the values of lower parameters known with a better accuracy.

6. The method presents a simple test for analyzing the compatibility of different models. The analysis consists of two steps. In the first step, the model parameters are expressed through cosmological parameters. The second step consists in finding the intervals of cosmological parameter changes that can be realized within the framework of the considered model. Since the cosmological parameters are universal, only in the case of a nonzero intersection of the obtained intervals, the models are compatible.

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