(MULTI–) FREQUENCY VARIATIONS OF STARS. SOME METHODS AND RESULTS

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ABSTRACT. Some algorithms and programs are described for determining the parameters of processes with constant and variable periods. Those are: FOUR-1 – periodogram analysis by using the least squares one–harmonic fit, FOUR-N corresponds to a number of harmonics, FOUR-T to a sine wave with a linear trend, FOUR-M to a number of waves with independent frequencies. Multishifted and mean weighted periodograms are discussed. PERMIN allows the determination of a best fit period from the moments of "characteristic events" only. The programs allows one to determine not only the parameters, but accuracy estimates and the "false alarm" probability as well. Some applications of these and some other methods to variable stars of different types are discussed.

Key words: Data reduction, Stars: binary, multiple, pulsating

Basic Equations

Although the method of the least squares is widely used (e.g. Whittaker and Robinson 1926, Anderson 1958), we briefly summarize the basic equations in the notation used for the applications below.

In general, the system of normal equations corresponding to the method of least squares may be written as

$$\sum_{\alpha=1}^{m} A_{\alpha\beta}\, C_\alpha \;=\; B_\beta, \qquad (1)$$

$$A_{\alpha\beta} \;=\; \sum_{k=1}^{n} f_\alpha(z_k)\; f_\beta(z_k), \qquad (2)$$

$$B_\beta \;=\; \sum_{k=1}^{n} x_k\; f_\beta(t_k)$$

and the coefficients $C_\alpha, \alpha = 1...m$ depend on the observations (x_k obtained at moments t_k as well as on the shape of the basic functions $f_\alpha(t)$.

Root mean squared error σ of the smoothing function

$$x_c(t) = \sum_{\alpha=1}^{m} C_\alpha f_\alpha(t) \qquad (3)$$

and its derivatives of the s^{th} derivative with respect to the parameter t may be determined as

$$\sigma^2[x_c^{[s]}(t)] = \sigma_*^2 \sum_{\alpha\beta=1}^{m} A^{-1}_{\alpha\beta} f_\alpha^{[s]}(t) f_\beta^{[s]}(t), \qquad (4)$$

where σ_*^2- is the "unit weight" error. The mathematical expectation of it is equal to

$$\sigma_*^2 = \frac{1}{n-m} \sum_{k=1}^{n} (x_k - x_c(t_k))^2, \qquad (5)$$

and $A^{-1}_{\alpha\beta}-$ is the matrix, inverse to $A_{\alpha\beta}$.

The moment of extremum t_e is a root of the equation $x_c^{[1]}(t_e) = 0$. An estimate of its r.m.s. value is

$$\sigma[t_e] = \frac{\sigma[x_c^{[1]}(t_e)]}{|x_c^{[2]}(t_e)|}. \qquad (6)$$

These expressions may be generalized for the case when the smoothing function depends on additional parameters D_γ, $\gamma = 1...p$. One may choose initial values and then determine differential corrections by using the system of equations

$$\sum_{\alpha=1}^{m} f_\alpha(z)\,\delta C_\alpha + \sum_{\gamma=1}^{p} \frac{\partial x_c}{\partial D_\gamma}\,\delta D_\gamma = x_k - x_c(t_k). \tag{7}$$

After iterating one may determine the $m+p$ parameters C_α, δD_γ and corresponding error estimates. Obviously, the derivatives

$$\frac{\partial x_c(t)}{\partial D_\gamma} = \sum_{\alpha=1}^{m} C_\alpha \left(\frac{\partial f_\alpha(t)}{\partial D_\gamma}\right)_{t=t_k} \tag{8}$$

may be determined by using coefficients C_α, determined from the m normal equations for fixed values of D_γ.

In other words, in the linearized model (7) one may formally write $\delta C_{m+\gamma} = \delta D_\gamma$ and $f_{m+\gamma} = \partial x_c(t)/\partial D_\gamma$ (computed according to Eq. (8)) suggesting that the number of the unknowns is $m' = m+p$. Error estimates may be computed similar to Eq. (4–6). It is important to point out that the error estimates of the coefficients C_α obtained from the models with m and $m+p$ unknowns are generally *different.*

In the least squares methods one compares the variance of the residuals with that of the initial observations. Thus as a basic function, we have used the statistics

$$S(f) = \frac{\sigma_C^2}{\sigma_O^2} = 1 - \frac{\sigma_{O-C}^2}{\sigma_O^2}, \tag{9}$$

where σ_O is the r.m.s. deviation of the "observations" O from the sample mean. C corresponds to "calculated" values and $O-C$ to the deviations of the "observed" values from the "calculated" ones.

If the values x_k are normally distributed, with the same mean and variance, uncorrelated random data (hereafter "random"), then the random variable S has a B-distribution

$$\rho(S) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\,\Gamma(\nu)}\, S^{\mu-1}\,(1-S)^{\nu-1} \tag{10}$$

(cf. Mardia and Zemroch 1978) with parameters $\mu = n_1/2$, $\nu = n_2/2$, where n_1 is the number of additional degrees of freedom used for the fit as compared with that used for determination of σ_O. The mathematical expectation of the mean value is $<S> = \mu/(\mu+\nu)$.

A brief discussion of the period search methods with application to some programs described here was presented by Andronov (1991a). A more detailed review and list of references is given in Andronov (1995).

One–harmonic fit (FOUR–1)

The model is

$$x_c(t) = C_1 + C_2 \sin\omega t + C_3 \cos wt, \tag{11}$$

thus the values of the basic functions at arguments t_k are $f_1(t_k) = 1$, $f_2 = \sin(\omega t_k)$, $f_3 = \cos(\omega t_k)$. Here $\omega = 2\pi f$, where f is trial frequency.

The "true" least squares fit (11) differs from the widely used approximations by Deeming (1975) and Lomb (1976).

To determine σ_0 one uses $n-1$ independent variables with the mean subtracted. For the one–frequency model two additional parameters C_2 and C_3 are determined, thus the number of the degrees of freedom are $n_1 = 2$ and $n_2 = n - 1 - n_1 = n - 3$ (cf. Andronov 1991):

$$\rho(S) = \frac{n-3}{2}(1-S)^{(n-5)/2} \tag{12}$$

with a corresponding mean value $<S> = 2/(n-1)$ The probability $Pr_1 = \mathrm{Prob}(S > S_0)$ of $S > S_0$ is equal to

$$Pr_1 = \int_{S_0}^{1} \rho(S)dS = (1-S_0)^{(n-3)/2}. \tag{13}$$

It may be noted that this expression differs significantly from the approximation $Pr_1 = \exp(-S_0/<S>)$ (e.g. Scargle 1982) which is usually used. This approximation in fact corresponds to the χ_2^2-distribution. However, the χ^2-approximation may be used only if one knows the general variance of the observations σ^2. The estimate σ_O^2 is used instead for the definition of $S(f)$. Thus the statistically justified distribution for the case of random normally distributed observations is described by the expressions mentioned above.

For many frequencies one has to estimate a "false alarm" probability

$$Pr = 1 - (1 - Pr_1)^K, \tag{14}$$

(Scargle 1982, Terebizh 1992) where K is the number of "independent frequencies" which may be estimated for n observations which are nearly equidistantly distributed in time as $K \approx (f_{max} - f_{min})/\Delta f$, where $\Delta f = n/((n-1)(t_n - t_1))$ for observations equidistantly distributed in time.

The spectral window is computed as

$$S_w(f) = \left(\frac{1}{n}\sum_{k=1}^{n}\cos\omega t_k\right)^2 + \left(\frac{1}{n}\sum_{k=1}^{n}\sin\omega t_k\right)^2 \quad (15)$$

(cf. Deeming 1975, Terebizh 1992). It is equal to unity for $\omega = 0$ and must be close to zero for "good" (nearly equidistant in time) observations. If the signal contains periodic components with frequencies f_{0j} and the spectral window has peaks at frequencies f_{wi} then a number of "alias" peaks at frequencies $|f_{0j} \pm f_{wi}|$ may be seen.

One may note that the periodograms of real observations often obey a power law: $S(f) \propto F^{-\gamma}$ indicating correlations between the subsequent data. For infinite data with "white noise" $\gamma = 0$, or "flicker noise" $\gamma = 1$, and for "random walks" $\gamma = 2$ (Terebizh 1992). Influence of the finite length of data runs on periodogram shapes is discussed by Andronov (1995). The power laws may be produced by several mechanisms – e.g. fractals, autoregressive processes, slow trends, and non-coherent oscillations.

The program FOUR–1 is arranged in the following way. First input file with 7 guidelines contains:

1. File name with the input data in a free format: two columns t_k, x_k separated by a blank space.

2. Output file 1 – results of the periodogram analysis determining the highest peak by using the differential corrections and describing all the peaks by fitting them by a parabola.

3. Output file name 2 - periodogram containing the columns: frequency, $S(f) + 0.6$, $S_w(f)$. Such format is convenient for drawing a periodogram using graphic editors. The arbitrary shift 0.6 is included to show in the same figure both the periodogram and the spectral window.

4. First trial frequency f_1.

5. Frequency step $\Delta f = \eta/(t_n - t_1)$ with $\eta \approx 0.10$ (cf. Kholopov 1971).

6. Number of trial frequencies.

7. Output file containing two columns with t_k and $x_k - x_c(t_k)$. One may use this file to compute a periodogram for these "prewhitened" $(O - C)$ observations.

The program is "non–stop", i.e. after finishing computation of the periodogram it reads from the file next 7 guidelines and starts again.

Multi–Harmonic Fit (FOUR–N)

For one–frequency signals with a complicated shape one may use basic functions ($f_1 = 1$, $f_{2j} = \sin(j\omega t)$, $f_{2j+1} = \cos(j\omega t)$, $j = 1...s$). For random data the number of degrees of freedom is $n_1 = 2s$, $n_2 = n - 1 - 2s$. After preliminary determination of the best-fit frequency one may use the method of differential corrections with

$$f_{2s+2}(t) = t\sum_{j=1}^{s}(C_{2j}\cos(j\omega t) - C_{2j+1}\sin(j\omega t))j. \quad (16)$$

An error estimate $\sigma[\omega] = \sigma_* \cdot (A^{-1}_{2s+2,2s+2})^{1/2}$. Obviously, $\sigma[f] = \sigma[\omega]/(2\pi)$ and $\sigma[P] = \sigma[f] \cdot f^{-2}$.

The problem is to choose an adequate number of harmonics s. In our program we choose the maximal number s_0 (usually 4–5) and compute periodograms for all $s \le s_0$. Than we choose a preliminary value of the frequency and use the method of differential corrections. If one will plot a $\sigma_*(s)$ diagram, one may see that σ_* decreases with s for small s and then it is nearly constant within error estimates. Thus one method is to determine the number s, after which the significant decrease of σ_* stops (cf. Terebizh 1992). One may compute the parameter

$$V_s = \frac{(n - 2s + 1)\sigma_*^2(s-1)}{(n - 2s - 1)\sigma_*^2(s)} - 1 \quad (17)$$

which (for random data) has the Fischer distribution with 2 and $(n - 2s - 1)$ degrees of freedom (cf. Mardia and Zemroch 1978). Thus one may choose a confidence level and determine s in the interval from 1 to s_0.

The r.m.s. value of the error estimate σ_{obs} of the smoothing function at the times of observations is defined as

$$\sigma_{obs}^2 = \frac{\sigma_*^2}{n} \sum_{k=1}^{n} \sum_{\alpha\beta=1}^{m} A_{\alpha\beta}^{-1} f_\alpha(t_k) f_\beta(t_k) = \frac{m}{n}\sigma_*^2. \tag{18}$$

To minimize the statistical error of the smoothing curve, one has to choose m minimizing the right side of this equation. However, for noisy signals, the value of σ_* decreases with m not very fast, thus one may formally prefer to use one–harmonic fit or even a constant mean value.

The same problem occurs for minimizing the r.m.s. error estimate σ_{phase} of the smoothing function at all phases:

$$\begin{aligned} \sigma_{phase}^2 &= \sigma_*^2 \cdot \frac{1}{P} \int_0^P \sum_{\alpha\beta=1}^{m} A_{\alpha\beta}^{-1} f_\alpha(t) f_\beta(t)\, dt = \\ &= \sigma_*^2 \left(A_{11}^{-1} + \frac{1}{2} \sum_{\alpha=2}^{m} A_{\alpha\alpha}^{-1} \right). \end{aligned} \tag{19}$$

Contrary to the error estimate of the moment of extremum (Eq.(6)), the error estimate $\sigma[U]$ of the asymmetry of the light curve defined as $U = (t_{max} - t_{min})/P$ may be computed by using the more complicated expression

$$\sigma^2[U] = P^{-2}\sigma_*^2 \sum_{\alpha\beta=1}^{m} Z_\alpha Z_\beta \tag{20}$$

Here $Z_\alpha = z(t_{max}) - z(t_{min})$ and

$$z_\alpha(t) = -\frac{\dot{f}_\alpha(t)}{\sum_{\varepsilon=1}^{m} C_\varepsilon \ddot{f}_\varepsilon(t)} \tag{21}$$

For multiharmonic fit, $z_1(t) = 0$,

$$\begin{aligned} z_{2j}(t) &= -j\cos(j\omega t)/(wY) \\ z_{2j+1}(t) &= j\sin(j\omega t)/(wY) \\ Y &= \sum_{j=1}^{s} j^2 (C_{2j}\sin(j\omega t) + C_{2j+1}\cos(j\omega t)) \end{aligned} \tag{22}$$

Multi–Frequency Fit (FOUR–M)

The basic functions are $f_1(t) = 1$, $f_{2j}(t) = \sin(\omega_j t)$, $f_{2j+1}(t) = \cos(\omega_j t)$, $j = 1...s$. For preliminary determination of the frequencies one may compute a grid of models with different combinations of the values of ω_j with frequency steps defined as for one–frequency models. The test function $S(f_1, ..., f_s)$ for random observations obeys $B-$ distribution with $\mu = s$ and $\nu = (n-1-2s)/2$. For more precise determinations of the parameters one may use the method of the differential corrections, so

$$f_{2s+1+j}(t) = (C_{2j}\cos(\omega_j t) - C_{2j+1}\sin(\omega_j t))t. \tag{23}$$

An alternate method is "prewhitening" (e.g. Terebizh 1992), in which one frequency models are applied to the initial observations. Then the best fit is subtracted from the data, the periodogram is recomputed for the residuals, the new wave is subtracted etc. This method is useful for preliminary determination of the frequencies. However, the method of differential corrections allows the determination of all the frequencies correctly.

A popular program to determine parameters of the multi–harmonic and multi–frequency fits was published by Breger (1990). Our program allows also one to obtain the corresponding error estimates.

An example of application of this code to the semiregular variable RX Boo was published by Andronov and Kudashkina (1988b).

Multi–Shift Fit (FOUR–S)

Such a model may be applied if the observations are subdivided into r separate runs and there may be run–to–run changes owed to the long–term variations of the object. Also there may be small shifts between the instrumental systems if the runs were obtained in different observatories.

The basic functions are the following: $f_\alpha(t_k) = 1$, if the observation belongs to the α^{th} run and 0 else. Other basic functions are sines and cosines similar to the discussed above. If the number of the frequencies used is s than $m = r + 2s$ and the parameters of the $B-$ distribution are $\mu = s$ and $\nu = (n - r - 2s)/2$.

This model was applied e.g. for 2–frequency fits of 15 runs of the cataclysmic variable TT Ari (Tremko et al. 1992) taking into account different weights of the individual observations.

Mean weighted periodograms

The mean weighted test function $S(f)$ calculated from the values S_i corresponding to the season No. i and the given trial frequency, by using the expression

$$S(f) = \frac{1}{n\sigma_O^2}\sum_{i=1}^{q} n_i\sigma_{Oi}^2 S_i(f),$$
$$n\sigma_O^2 = \sum_{i=1}^{q} n_i\sigma_{Oi}^2, \quad n = \sum_{i=1}^{q} n_i. \qquad (24)$$

where n_i is the number of observations in the i^{th} season, and σ_{Oi}^2 - is the variance in the same run. This model (24) takes into account the possibility that the variations of the mean brightness a_i occur, as well as the amplitude and the initial phase. The variations of the values σ_{Oi} are assumed to be attributed to the apparent random changes from run to run, but the general dispersion σ_0 is the same for all q runs. For random data the test function $S(f)$ has the "$B-$type" probability distribution function with $n_1 = 2q$ and $n_2 = n - 3q$ degrees of freedom (see Andronov et al. 1992 for details).

The mean value of $S(f)$ for randomly distributed observations is $< S > = 2q/(n-q)$.

Moments of the Characteristic Events (PERMIN)

Andronov (1991, 1993) proposed the method and studied the statistical properties of test functions which are more complicated compared with that for normally distributed observations. The trial frequencies are chosen to be $f_i = i/(t_n - t_1)$ with corresponding least squares correction for all i. The program is computationally efficient because the frequency step is $\approx$ 10 times larger than that in the above mentioned cases.

A more complicated method was proposed by Dumont et al. (1978). Their method is compared with the least squares method in Andronov (1988).

Some dwarf nova stars show fast period changes from one value to another. These data were fitted by hyperbolic functions (Andronov and Shakun 1990). Similar changes occured in the semi–regular variable AF Cyg (Andronov and Chernyshova 1989).

Periodic Variations of O–C (FOUR–T)

This method may be applied for the moments of the characteristic events, but mathematically it is the same as the one–harmonic model with trend, so

$$\delta T_0 + E_k\delta P + C_3\cos(\omega_E E_k) + \\ +C_4\sin(\omega_E E_k) = (O-C)_k \qquad (25)$$

Here $C_1 = \delta T_0$, $C_2 = \delta P$ are differential corrections to the initial epoch and period, respectively, and $(O-C)_k$ are deviations of the observed times from the ones "calculated" using a linear ephemeris $T_k = T_0 + PE_k$. The trial period $P_E = 2\pi/\omega_E$. The test function $S(f)$ for random data obeys the $B-$ distribution (10), but for n equal to the number of observations minus unity. After determination of the preliminary value of ω_E one may correct it by the method of differential corrections.

This program was applied to study various phenomena, e.g. the orientation changes of the white dwarf in the magnetic binaries AM Her (Andronov et al. 1982) and QQ Vul (Andronov and Fuhrmann 1987), the Blazhko effect in TT Cnc (Andronov et al. 1985), the presence of the third body in the eclipsing variable AK Her (Andronov et al. 1989).

"Smoothing the Smoothing Cubic Splines"

The method was proposed by Andronov (1987) and applied e.g. to the exotic binary V 361 Lyr (Andronov and Richter 1987) with a hot spot between the components, to a polar MR Ser (Andronov et al. 1992) and to a number of the Mira–type stars (Andronov et al. 1988ac, 1992a).

Autocorrelation Function Analysis of the Detrended Data of Finite Length

Influence of the finite length of the data run and subtraction of the least squares fit in general form was discussed by Andronov (1994). If the unbiased ACF is ρ_u, the biased one is r_u, one may write the mathematical expectation

$$r_u = R_u/R_0,$$

$$R_u = \sum_{i=0}^{n-1} Z_{iu}\rho_i. \quad (26)$$

The matrix Z_{iu} is a degenerate, thus one may not restore the unbiased ACF from observations. The only way involves the modeling of ρ_i and the determination of the model parameters by fitting the observed ACF with a computed one r_u. Thus the restored ACF is model–dependent.

One may note that the $n \times n$ matrix Z_{iu} depends on the basic functions used to determine the removed trend and on the run length n. Computational time for $n = 256$ and cubic polynomial is ≈ 60 hours using a 33 MHz PC–486. Such a matrix is to be computed for a desired regime and then stored as a file for further data fits.

Although the ACF may be used to study periodic signals, it is much more useful to determine parameters of the autoregressive processes which allows estimates of the contribution of the uncorrelated noise.

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